Formula List of
Applications and Interpretation
Higher Level
for IBDP Mathematics

Analysis & Approaches Standard Level	**Analysis & Approaches Higher Level**
Applications & Interpretation Standard Level	**Applications & Interpretation Higher Level**

1 Standard Form

✓ Standard Form:

A number in the form $(\pm)a\times10^{k}$, where $1\leq a<10$ and k is an integer

2 Approximation and Error

✓ Summary of rounding methods:

2.71828	Correct to 3 significant figures	Correct to 3 decimal places
Round off	2.7**2**	2.71**8**

✓ Consider a quantity measured as Q and correct to the nearest unit d:

$\frac{1}{2}d$: Maximum absolute error

$Q-\frac{1}{2}d\leq A<Q+\frac{1}{2}d$: Range of the actual value A

$Q-\frac{1}{2}d$: Lower bound (Least possible value) of A

$Q+\frac{1}{2}d$: Upper bound of A

$\frac{\text{Maximum absolute error}}{Q}\times100\%$: Percentage error

3 Functions

- ✓ The function $y = f(x)$:
  1. $f(a)$: Functional value when $x = a$
  2. Domain: Set of values of x
  3. Range: Set of values of y

- ✓ Properties of rational function $y = \frac{ax+b}{cx+d}$:
  1. $y = \frac{1}{x}$: Reciprocal function
  2. $y = \frac{a}{c}$: Horizontal asymptote
  3. $x = -\frac{d}{c}$: Vertical asymptote

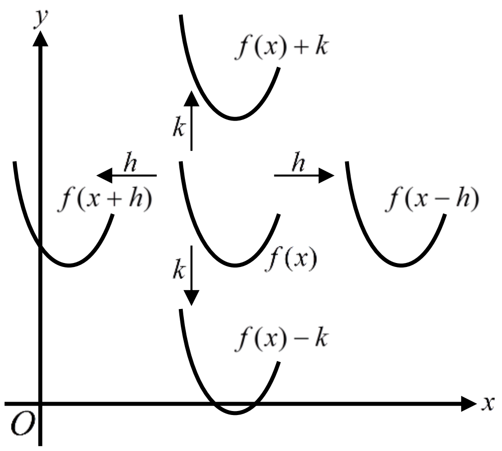
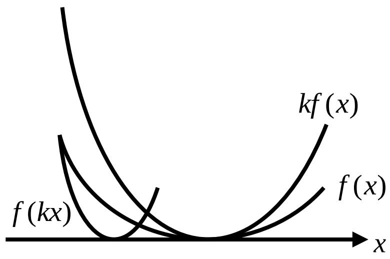
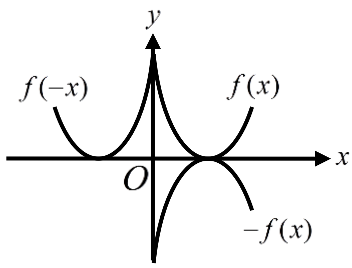
- ✓ $f \circ g(x) = f(g(x))$: Composite function when $g(x)$ is substituted into $f(x)$

- ✓ Steps of finding the inverse function $y = f^{-1}(x)$ of $f(x)$:
  1. Start from expressing y in terms of x
  2. Interchange x and y
  3. Make y the subject in terms of x

- ✓ Properties of $y = f^{-1}(x)$:
  1. $f(f^{-1}(x)) = f^{-1}(f(x)) = x$
  2. The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about $y = x$

- ✓ $f^{-1}(x)$ exists only when $f(x)$ is one-to-one in the restricted domain

✓ Summary of transformations:

	$f(x)\to f(x)+k$: Translate upward by k units
	$f(x)\to f(x)-k$: Translate downward by k units
	$f(x)\to f(x+h)$: Translate to the left by h units
	$f(x)\to f(x-h)$: Translate to the right by h units
	$f(x)\to kf(x)$: Vertical stretch of scale factor k
	$f(x)\to f(kx)$: Horizontal compression of scale factor k
	$f(x)\to -f(x)$: Reflection about the x-axis
	$f(x)\to f(-x)$: Reflection about the y-axis

✓ Variations:

1. $y=kx, k\neq 0$: y is directly proportional to x
2. $y=\frac{k}{x}, k\neq 0$: y is inversely proportional to x

4 Quadratic Functions

✓ General form $y = ax^2 + bx + c$, where $a \neq 0$:

$a > 0$	The graph opens upward
$a < 0$	The graph opens downward
c	y-intercept
$h = -\dfrac{b}{2a}$	x-coordinate of the vertex
$k = ah^2 + bh + c$	y-coordinate of the vertex
	Extreme value of y
$x = h$	Equation of the axis of symmetry

✓ Other forms:

1. $y = a(x-h)^2 + k$: Vertex form
2. $y = a(x-p)(x-q)$: Factored form with x-intercepts p and q

✓ $h = -\dfrac{b}{2a} = \dfrac{p+q}{2}$

✓ The x-intercepts of the quadratic function $y = ax^2 + bx + c$ are the roots of the corresponding quadratic equation $ax^2 + bx + c = 0$

5 Exponential and Logarithmic Functions

✓ $y = a^x$: Exponential function, where $a \neq 1$

✓ $y = \log_a x$: Logarithmic function, where $a > 0$

✓ $y = \log x = \log_{10} x$: Common Logarithmic function

✓ $y = \ln x = \log_e x$: Natural Logarithmic function, where $e = 2.71828...$ is an exponential number

✓ Properties of the graphs of $y = a^x$:

$a > 1$	$0 < a < 1$
y-intercept $= 1$	
y increases as x increases	y decreases as x increases
y tends to zero as x tends to negative infinity	y tends to zero as x tends to positive infinity
Horizontal asymptote: $y = 0$	

✓ Laws of logarithm, where a, b, c, p, q, $x > 0$:

1. $x = a^y \Leftrightarrow y = \log_a x$
2. $\log_a 1 = 0$
3. $\log_a a = 1$
4. $\log_a p + \log_a q = \log_a pq$
5. $\log_a p - \log_a q = \log_a \frac{p}{q}$
6. $\log_a p^n = n \log_a p$
7. $\log_b a = \frac{\log_c a}{\log_c b}$

✓ $f(x) = \frac{L}{1 + Ce^{-kx}}$: Logistic function, where L, C and k are positive constants

✓ Semi-log model:

1. $y = k \cdot a^x \Leftrightarrow \ln y = (\ln a)x + \ln k$: Semi-log model
2. $\ln a$: Gradient of the straight line graph on $\ln y$-x plane
3. $\ln k$: Vertical intercept of the straight line graph on $\ln y$-x plane

✓ Log-log models:

1. $y = k \cdot x^n \Leftrightarrow \ln y = n \ln x + \ln k$: Log-log model
2. n: Gradient of the straight line graph on $\ln y$-$\ln x$ plane
3. $\ln k$: Vertical intercept of the straight line graph on $\ln y$-$\ln x$ plane

6 Systems of Equations

✓ $\begin{cases} ax+by=c \\ dx+ey=f \end{cases}$: 2×2 system

✓ $\begin{cases} ax+by+cz=d \\ ex+fy+gz=h \\ ix+jy+kz=l \end{cases}$: 3×3 system

✓ The above systems can be solved by PlySmlt2 in TI-84 Plus CE

7 Arithmetic Sequences

✓ Properties of an arithmetic sequence u_n:

1. u_1: First term
2. $d=u_2-u_1=u_n-u_{n-1}$: Common difference
3. $u_n=u_1+(n-1)d$: General term (n th term)
4. $S_n=\frac{n}{2}[2u_1+(n-1)d]=\frac{n}{2}[u_1+u_n]$: The sum of the first n terms

✓ $\sum_{r=1}^{n} u_r = u_1+u_2+u_3+\cdots+u_{n-1}+u_n$: Summation sign

8 Geometric Sequences

✓ Properties of a geometric sequence u_n:

1. u_1: First term
2. $r=u_2\div u_1=u_n\div u_{n-1}$: Common ratio
3. $u_n=u_1\times r^{n-1}$: General term (n th term)
4. $S_n=\frac{u_1(1-r^n)}{1-r}$: The sum of the first n terms

✓ $S_\infty = \frac{u_1}{1-r}$: The sum to infinity of a geometric sequence u_n, given that $-1 < r < 1$

9 Financial Mathematics

✓ Compound Interest:

PV : Present value

$r\%$: Interest rate per annum (per year)

n: Number of years

k: Number of compounded periods in one year

$FV = PV\left(1+\frac{r}{100k}\right)^{kn}$: Future value

$I = FV - PV$: Interest

✓ Inflation:

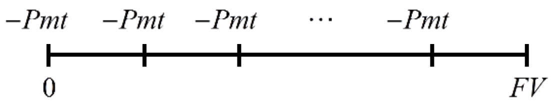
$i\%$: Inflation rate

$R\%$: Interest rate compounded yearly

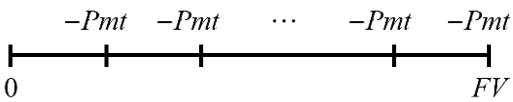
$(R-i)\%$: Real rate

✓ Annuity:

1. Payments at the beginning of each year

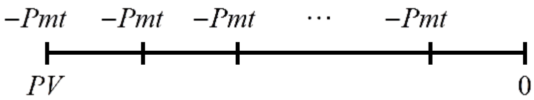


2. Payments at the end of each year

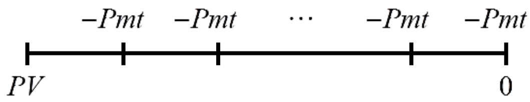


✓ Amortization:

1. Payments at the beginning of each year



2. Payments at the end of each year



10 Coordinate Geometry

✓ Consider the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on a x - y plane:

1. $m = \frac{y_2 - y_1}{x_2 - x_1}$: Slope of PQ
2. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$: Distance between P and Q
3. $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$: Mid-point of PQ

✓ Consider the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on a x - y - z plane:

1. z -axis: The axis perpendicular to the x - y plane
2. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$: Distance between P and Q
3. $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$: Mid-point of PQ

✓ Forms of straight lines with slope m and y -intercept c :

1. $y = mx + c$: Slope-intercept form
2. $Ax + By + C = 0$: General form

✓ Ways to find the x -intercept and the y -intercept of a line:

1. Substitute $y = 0$ and make x the subject to find the x -intercept
2. Substitute $x = 0$ and make y the subject to find the y -intercept

11 Voronoi Diagrams

✓ Elements in Voronoi Diagrams:

Site: A given point

Cell of a site: A collection of points which is closer to the site than other sites

Boundary: A line dividing the cells

Vertex: An intersection of boundaries

✓ Related problems:

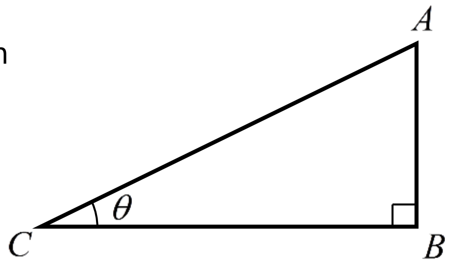
1. Nearest neighbor interpolation
2. Incremental algorithm
3. Toxic waste dump problem

12 Trigonometry

✓ Consider a right-angled triangle ABC:

$\mathrm{AB}^2 + \mathrm{BC}^2 = \mathrm{AC}^2$: Pythagoras' Theorem

$$\begin{cases} \sin\theta = \dfrac{\mathrm{AB}}{\mathrm{AC}} \\ \cos\theta = \dfrac{\mathrm{BC}}{\mathrm{AC}} \\ \tan\theta = \dfrac{\mathrm{AB}}{\mathrm{BC}} \end{cases}$$: Trigonometric ratios



✓ Properties of a general trigonometric function $y = A\sin B(x - C) + D$:

1. $A = \dfrac{y_{\max} - y_{\min}}{2}$: Amplitude
2. $B = \dfrac{360^\circ}{\text{Period}}$ or $\dfrac{2\pi}{\text{Period}}$
3. $D = \dfrac{y_{\max} + y_{\min}}{2}$
4. C can be found by substitution of a point on the graph

✓ Properties of graphs of trigonometric functions:

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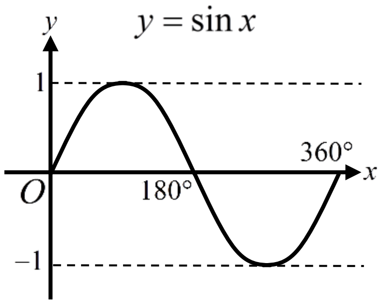
<tr><td rowspan="4">
</td><td>1.</td><td>Amplitude $= 1$</td></tr>
<tr><td>2.</td><td>Period $= 360^\circ$ or 2π</td></tr>
<tr><td>3.</td><td>$-1 \le \sin x \le 1$</td></tr>
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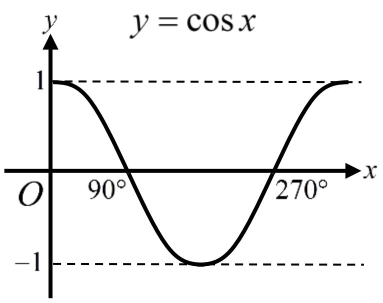
<tr><td rowspan="4">
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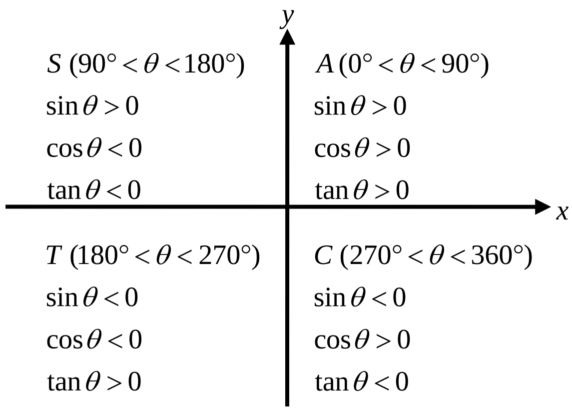
<tr><td></td><td></td></tr>
</table>

✓ Trigonometric identities:

1. $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$

2. $\sin^2\theta + \cos^2\theta \equiv 1$

✓ ASTC diagram



13 2-D Trigonometry

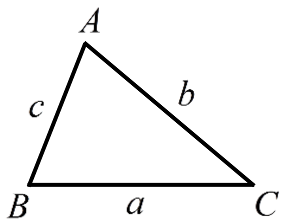
✓ Consider a triangle ABC:

1. $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$ or $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$: Sine rule

2. $a^2 = b^2 + c^2 - 2bc\cos A$

   or $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$: Cosine rule

3. $\dfrac{1}{2}ab\sin C$: Area of the triangle ABC

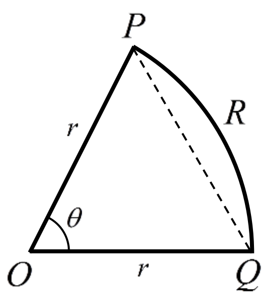


✓ Consider a sector $OPRQ$ with centre O, radius r and $\angle POQ = \theta^\circ$:

$2\pi r \times \dfrac{\theta^\circ}{360^\circ}$: Arc length PRQ

$\pi r^2 \times \dfrac{\theta^\circ}{360^\circ}$: Area of the sector $OPRQ$

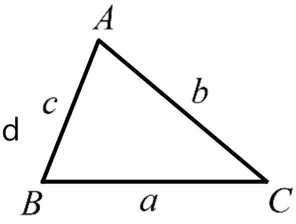
$\pi r^2 \times \dfrac{\theta^\circ}{360^\circ} - \dfrac{1}{2}r^2\sin\theta^\circ$: Area of the segment PRQ



✓ Consider a triangle ABC:

$\frac{\sin A}{a}=\frac{\sin B}{b}$ or $\frac{a}{\sin A}=\frac{b}{\sin B}$: Sine rule

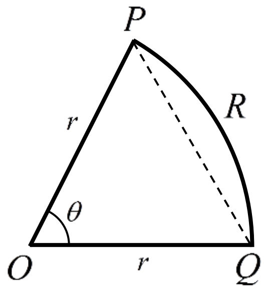
Note: The ambiguous case exists if two sides and an angle are known, and the angle is opposite to the shorter known side



✓ $\frac{x^{\circ}}{180^{\circ}}=\frac{y \text{ rad}}{\pi \text{ rad}}$: Method of conversions between degree and radian

✓ Consider a sector $OPRQ$ with centre O, radius r and $\angle POQ=\theta$ in radian:

1. $r\theta$: Arc length PQ
2. $\frac{1}{2}r^2\theta$: Area of the sector $OPRQ$
3. $\frac{1}{2}r^2(\theta-\sin\theta)$: Area of the segment PRQ



14 Areas and Volumes

✓ For a cube of side length l:

1. $6l^2$: Total surface area
2. l^3: Volume

✓ For a cuboid of side lengths a, b and c:

1. $2(ab+bc+ac)$: Total surface area
2. abc: Volume

✓ For a prism of height h and cross-sectional area A:

1. Ah: Volume

✓ For a cylinder of height h and radius r:

1. $2\pi r^2+2\pi rh$: Total surface area
2. $2\pi rh$: Lateral surface area
3. $\pi r^2 h$: Volume

✓ For a pyramid of height h and base area A:

1. $\frac{1}{3}Ah$: Volume

✓ For a circular cone of height h and radius r:

1. $l=\sqrt{r^2+h^2}$: Slant height
2. $\pi r^2+\pi rl$: Total surface area
3. πrl: Curved surface area
4. $\frac{1}{3}\pi r^2 h$: Volume

✓ For a sphere of radius r:

1. $4\pi r^2$: Total surface area
2. $\frac{4}{3}\pi r^3$: Volume

✓ For a hemisphere of radius r:

1. $3\pi r^2$: Total surface area
2. $2\pi r^2$: Curved surface area
3. $\frac{2}{3}\pi r^3$: Volume

15 Complex Numbers

✓ Terminologies of complex numbers:

$\mathrm{i}=\sqrt{-1}$: Imaginary unit

$z=a+b\mathrm{i}$: Complex number in Cartesian form

a: Real part of z

b: Imaginary part of z

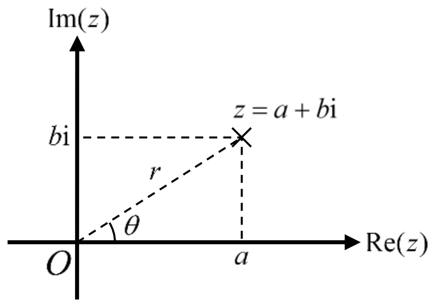
$z^*=a-b\mathrm{i}$: Conjugate of $z=a+b\mathrm{i}$

$|z|=r=\sqrt{a^2+b^2}$: Modulus of $z=a+b\mathrm{i}$

$\arg(z)=\theta=\arctan\frac{b}{a}$: Argument of $z=a+bi$

✓ Properties of Argand diagram:

1. Real axis: Horizontal axis
2. Imaginary axis: Vertical axis



✓ Forms of complex numbers:

1. $z = a + b\mathrm{i}$: Cartesian form
2. $z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{cis}\theta$: Modulus-argument form
3. $z = re^{\mathrm{i}\theta}$: Euler form

✓ Properties of moduli and arguments of complex numbers z_1 and z_2 :

1. $|z_1 z_2| = |z_1||z_2|$
2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
3. $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
4. $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

✓ If $z = a + b\mathrm{i}$ is a root of the polynomial equation $p(z) = 0$, then $z^* = a - b\mathrm{i}$ is also a root of $p(z) = 0$

16 Matrices

✓ Terminologies of matrices:

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$: A $m \times n$ matrix with m rows and n columns

a_{ij} : Element on the i th row and the j th column

$$\mathbf{I}=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$: Identity matrix

$$\mathbf{0}=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$: Zero matrix

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$: Diagonal matrix

$|\mathbf{A}|=\det(\mathbf{A})$: Determinant of $\mathbf{A}$

$\mathbf{A}$ is non-singular if $\det(\mathbf{A}) \neq 0$

$\mathbf{A}^{-1}$: Inverse of $\mathbf{A}$

$\mathbf{A}^{-1}$ exists if $\mathbf{A}$ is non-singular

✓ For any 2×2 square matrices $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

1. $|\mathbf{A}|=\det(\mathbf{A})=ad-bc$: Determinant of $\mathbf{A}$

2. $\mathbf{A}^{-1}=\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$: Inverse of $\mathbf{A}$

✓ Operations of matrices:

1. $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \pm \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} \pm b_{11} & \cdots & a_{1n} \pm b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & \cdots & a_{mn} \pm b_{mn} \end{pmatrix}$

2. $k\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & \ddots & \vdots \\ ka_{m1} & \cdots & ka_{mn} \end{pmatrix}$

3. $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$: The element on the i th row and the j th column of $\mathbf{C} = \mathbf{AB} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nk} \end{pmatrix}$, where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are $m \times n$, $n \times k$ and $m \times k$ matrices respectively

✓ A 2×2 system $\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$ can be expressed as $\mathbf{AX} = \mathbf{B}$, where $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ can be solved by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{pmatrix} a & b \\ d & e \end{pmatrix}^{-1}\begin{pmatrix} c \\ f \end{pmatrix}$

✓ A 3×3 system $\begin{cases} ax + by + cz = d \\ ex + fy + gz = h \\ ix + jy + kz = l \end{cases}$ can be expressed as $\mathbf{AX} = \mathbf{B}$, where $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ can be solved by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{pmatrix} a & b & c \\ e & f & g \\ i & j & k \end{pmatrix}^{-1}\begin{pmatrix} d \\ h \\ l \end{pmatrix}$

✓ Eigenvalues and eigenvectors of $\mathbf{A}$:

1. $\det(\mathbf{A} - \lambda\mathbf{I})$: Characteristic polynomial of $\mathbf{A}$
2. Solution(s) of $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$: Eigenvalue(s) of $\mathbf{A}$
3. $\mathbf{v}$: Eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, which satisfies $\mathbf{Av} = \lambda\mathbf{v}$

✓ Diagonalization of $\mathbf{A}$:

1. $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$: Diagonal matrix of the eigenvalues of $\mathbf{A}$
2. $\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}$: A matrix of the eigenvectors of $\mathbf{A}$
3. $\mathbf{A} = \mathbf{VDV}^{-1} \Rightarrow \mathbf{A}^n = \mathbf{VD}^n\mathbf{V}^{-1}$

✓ Two-dimensional transformation matrices:

1. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$: Reflection about the x-axis

2. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$: Reflection about the y-axis

3. $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$: Reflection about the line $y = mx$, where $m = \tan\theta$

4. $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$: Vertical stretch with scale factor k

5. $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$: Horizontal stretch with scale factor k

6. $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$: Enlargement about the origin with scale factor k

7. $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$: Rotation with positive angle θ anticlockwise about the origin

8. $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$: Rotation with positive angle θ clockwise about the origin

9. Area of the image $= |\det(T)| \cdot$ Area of the object, where T is the transformation matrix

17 Vectors

✓ Terminologies of vectors:

$\overrightarrow{\mathrm{AB}}$: Vector of length AB with initial point A and terminal point B

$\overrightarrow{\mathrm{OP}}$: Position vector of P, where O is the origin

$\left|\overrightarrow{\mathrm{AB}}\right|$: Magnitude (length) of $\overrightarrow{\mathrm{AB}}$

$\hat{\mathbf{v}} = \dfrac{1}{|\mathbf{v}|}\mathbf{v}$: Unit vector parallel to $\mathbf{v}$, with $|\hat{\mathbf{v}}| = 1$

$\mathbf{0}$: Zero vector

$\mathbf{i}$: Unit vector along the positive x-axis

$\mathbf{j}$: Unit vector along the positive y-axis

$\mathbf{k}$: Unit vector along the positive z-axis

✓ A vector $\mathbf{v}$ can be expressed as $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ or $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

✓ Properties of vectors:

1. $\overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{OB}} - \overrightarrow{\mathrm{OA}}$
2. $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \pm \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 \pm v_1 \\ u_2 \pm v_2 \\ u_3 \pm v_3 \end{pmatrix}$
3. $\mathbf{v}$ and $k\mathbf{v}$ are in the same direction if $k > 0$
4. $\mathbf{v}$ and $k\mathbf{v}$ are in opposite direction if $k < 0$
5. $k\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} kv_1 \\ kv_2 \\ kv_3 \end{pmatrix}$

✓ Properties of the scalar product $\mathbf{u} \cdot \mathbf{v}$ of $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$:

1. $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = |\mathbf{u}||\mathbf{v}|\cos\theta$
2. $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$
3. $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$
4. $\mathbf{u}$ and $\mathbf{v}$ are in the same direction if $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|$
5. $\mathbf{u}$ and $\mathbf{v}$ are in opposite direction if $\mathbf{u} \cdot \mathbf{v} = -|\mathbf{u}||\mathbf{v}|$
6. $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if $\mathbf{u} \cdot \mathbf{v} = 0$
7. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
8. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$

- ✓ Properties of the vector product $\mathbf{u}\times\mathbf{v}$ of $\mathbf{u}=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$:

1. $\mathbf{u}\times\mathbf{v}=\begin{pmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{pmatrix}=|\mathbf{u}||\mathbf{v}|\sin\theta\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}//(\mathbf{u}\times\mathbf{v})$
2. $\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=\mathbf{0}$
3. $\mathbf{i}\times\mathbf{j}=\mathbf{k}$, $\mathbf{j}\times\mathbf{k}=\mathbf{i}$ and $\mathbf{k}\times\mathbf{i}=\mathbf{j}$
4. $\mathbf{j}\times\mathbf{i}=-\mathbf{k}$, $\mathbf{k}\times\mathbf{j}=-\mathbf{i}$ and $\mathbf{i}\times\mathbf{k}=-\mathbf{j}$
5. $\mathbf{u}$ and $\mathbf{v}$ are parallel if $\mathbf{u}\times\mathbf{v}=0$
6. $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if $|\mathbf{u}\times\mathbf{v}|=|\mathbf{u}||\mathbf{v}|$
7. $\mathbf{u}\times\mathbf{v}=-(\mathbf{v}\times\mathbf{u})$

- ✓ The area of the parallelogram with adjacent sides $\vec{AB}$ and $\vec{AD}$ is $\left|\vec{AB}\times\vec{AD}\right|$

- ✓ The area of the triangle with adjacent sides $\vec{AB}$ and $\vec{AD}$ is $\frac{1}{2}\left|\vec{AB}\times\vec{AD}\right|$

- ✓ Forms of the straight line with fixed point $A(a_1,a_2,a_3)$ and direction vector $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$:

1. $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+t\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$, $t\in\mathbb{R}$
2. $\begin{cases}x=a_1+b_1t\\y=a_2+b_2t\\z=a_3+b_3t\end{cases}$: Parametric form

- ✓ Vector components:

1. $\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$: Vector component of $\mathbf{u}$ parallel to $\mathbf{v}$
2. $\frac{|\mathbf{u}\times\mathbf{v}|}{|\mathbf{v}|}$: Vector component of $\mathbf{u}$ perpendicular to $\mathbf{v}$

18 Graph Theory

✓ Terminologies of graphs:
Vertex: A point on a graph
Edge: Arcs that connect vertices
Walk: A sequence of edges
Path: A sequence of edges that passes through any vertex and any edge at most once
Degree of a vertex: Number of edges connecting the vertex
Connected graph: A graph that there exists at least one walk between any two vertices
Unconnected graph: A graph that there exist at least two vertices that there is no walk between them
Subgraph of a graph: A collection of some edges and vertices of the original graph
Loop: An edge that starts and ends at the same vertex
Simple graph: A graph that has no loops and no multiple edges connecting the same pair of vertices
Multiple graph: A graph that has multiple edges connecting at least one pair of vertices
Cycle: A path that the starting vertex is the end vertex
Tree: A connected graph with no cycles
Spanning tree: A tree that connects all vertices in the graph

✓ Directed graphs:
1. Directed graph: A graph that all edges are assigned with directions
2. In-degree of a vertex: Number of edges connecting and pointing towards the vertex
3. Out-degree of a vertex: Number of edges connecting and pointing away from the vertex

✓ Adjacency matrix $\mathbf{M}$ of a graph with n vertices:

1. $n \times n$: Order of $\mathbf{M}$
2. The entry $m_{ij} = 1$ if there is an edge connecting the vertex i and the vertex j, and $m_{ij} = 0$ if otherwise
3. $\mathbf{M}^p$ shows the number of walks of length p in the graph
4. $\sum_{r=1}^{p} \mathbf{M}^r$ shows the number of walks of length less than or equal to p in the graph
5. The column sum of a transition matrix of a directed graph must be equal to 1

✓ Algorithms of finding minimum spanning trees:

1. Kruskal's algorithm
2. Prim's algorithm

✓ Eulerian trails and circuits:

1. Trail: A sequence of edges that passes through any edge at most once
2. Circuit: A trail that the starting vertex is the end vertex
3. Eulerian trail: A trail that passes through all edges of a graph
4. Eulerian circuit: A circuit that passes through all edges of a graph
5. An Eulerian trail exists if there exists two and only two vertices of odd degree
6. An Eulerian circuit exists if all vertices are of even degree
7. Chinese postman problem can be used to find the route of minimum weight that covers all edges of a graph

✓ Hamiltonian paths and cycles:

1. Complete graph: A graph that there exists an edge for any pair of two vertices
2. Hamiltonian path: A path that passes through all vertices of a graph
3. Hamiltonian cycle: A cycle that passes through all vertices of a graph

✓ Travelling Salesman problem:

1. Travelling Salesman problem can be used to find the cycle of minimum weight that passes through all vertices of a graph
2. Nearest neighbour algorithm can be used to find the upper bound of the solution of a travelling salesman problem
3. Deleted vertex algorithm can be used to find the lower bound of the solution of a travelling salesman problem

19 Differentiation

✓ $\frac{\mathrm{d}y}{\mathrm{d}x}=f'(x)$: Derivative of the function $y=f(x)$ (First derivative)

✓ Rules of differentiation:

1. $f(x)=x^n \Rightarrow f'(x)=nx^{n-1}$
2. $f(x)=p(x)+q(x) \Rightarrow f'(x)=p'(x)+q'(x)$
3. $f(x)=cp(x) \Rightarrow f'(x)=cp'(x)$

✓ Relationship between graph properties and the derivatives:

1. $f'(x)>0$ for $a\le x\le b$: $f(x)$ is increasing in the interval
2. $f'(x)<0$ for $a\le x\le b$: $f(x)$ is decreasing in the interval
3. $f'(a)=0$: $(a,\ f(a))$ is a stationary point of $f(x)$
4. $f'(a)=0$ and $f'(x)$ changes from positive to negative at $x=a$: $(a,\ f(a))$ is a maximum point of $f(x)$
5. $f'(a)=0$ and $f'(x)$ changes from negative to positive at $x=a$: $(a,\ f(a))$ is a minimum point of $f(x)$

✓ Tangents and normals:

1. $f'(a)$: Slope of tangent at $x=a$
2. $\frac{-1}{f'(a)}$: Slope of normal at $x=a$
3. $y-f(a)=f'(a)(x-a)$: Equation of tangent at $x=a$
4. $y-f(a)=\left(\frac{-1}{f'(a)}\right)(x-a)$: Equation of normal at $x=a$

✓ Derivatives of a function $y=f(x)$:

1. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=f''(x)$: Second derivative
2. $\frac{\mathrm{d}^ny}{\mathrm{d}x^n}=f^{(n)}(x)$: n-th derivative

✓ More rules of differentiation:

<table>
<tr><td>$f(x)=\sin x \Rightarrow f'(x)=\cos x$</td><td>$f(x)=p(q(x)) \Rightarrow f'(x)=p'(q(x))\cdot q'(x)$</td></tr>
<tr><td>$f(x)=\cos x \Rightarrow f'(x)=-\sin x$</td><td rowspan="2">$f(x)=p(x)q(x)$
$\Rightarrow f'(x)=p'(x)q(x)+p(x)q'(x)$</td></tr>
<tr><td>$f(x)=\tan x \Rightarrow f'(x)=\dfrac{1}{\cos^2 x}$</td></tr>
<tr><td>$f(x)=e^x \Rightarrow f'(x)=e^x$</td><td rowspan="2">$f(x)=\dfrac{p(x)}{q(x)}$
$\Rightarrow f'(x)=\dfrac{p'(x)q(x)-p(x)q'(x)}{(q(x))^2}$</td></tr>
<tr><td>$f(x)=\ln x \Rightarrow f'(x)=\dfrac{1}{x}$</td></tr>
</table>

✓ $f''(a)=0$ and $f''(x)$ changes sign at $x=a$: $(a,\ f(a))$ is a point of inflexion of $f(x)$

✓ $\dfrac{\mathrm{d}N}{\mathrm{d}t}=\dfrac{\mathrm{d}N}{\mathrm{d}x}\cdot\dfrac{\mathrm{d}x}{\mathrm{d}t}$: Rate of change of N with respect to the time t

✓ Tests for optimization:

1. First derivative test
2. Second derivative test

✓ Applications in kinematics:

1. $s(t)$: Displacement with respect to the time t
2. $v(t)=s'(t)$: Velocity
3. $a(t)=v'(t)$: Acceleration

20 Integration and Trapezoidal Rule

✓ Integrals of a function $y=f(x)$:

1. $\int f(x)\mathrm{d}x$: Indefinite integral of $f(x)$
2. $\int_a^b f(x)\mathrm{d}x$: Definite integral of $f(x)$ from a to b

✓ Rules of integration:

1. $\int x^n \mathrm{d}x = \frac{1}{n+1}x^{n+1} + C$
2. $\int (p'(x) + q'(x))\mathrm{d}x = p(x) + q(x) + C$
3. $\int cp'(x)\mathrm{d}x = cp(x) + C$

✓ $\int_a^b f(x)\mathrm{d}x$: Area under the graph of $f(x)$ and above the x-axis, between $x = a$ and $x = b$, where $f(x) \geq 0$

✓ Trapezoidal Rule:
a, b ($a < b$): End points
n: Number of intervals
$h = \frac{b-a}{n}$: Interval width
$\int_a^b f(x)\mathrm{d}x$ can be estimated by $\frac{1}{2}h[f(x_0) + f(x_n) + 2(f(x_1) + f(x_2) + \ldots + f(x_{n-1}))]$

✓ Estimation by Trapezoidal Rule:

1. The estimation overestimates if the estimated value is greater than the actual value of $\int_a^b f(x)\mathrm{d}x$
2. The estimation underestimates if the estimated value is less than the actual value of $\int_a^b f(x)\mathrm{d}x$

✓ More rules of integration:

$\int \sin x\mathrm{d}x = -\cos x + C$	$\int e^x\mathrm{d}x = e^x + C$
$\int \cos x\mathrm{d}x = \sin x + C$	$\int \frac{1}{x}\mathrm{d}x = \ln x + C$
$\int \frac{1}{\cos^2 x}\mathrm{d}x = \tan x + C$	Integration by substitution
$\int_a^b f'(x)\mathrm{d}x = [f(x)]_a^b = f(b) - f(a)$	

✓ Areas on x-y plane, between $x = a$ and $x = b$:

1. $\int_a^b |f(x)|\mathrm{d}x$: Area between the graph of $f(x)$ and the x-axis
2. $\int_a^b |f(x) - g(x)|\mathrm{d}x$: Area between the graph of $f(x)$ and the graph of $g(x)$

✓ Areas on x - y plane, between $y=c$ and $y=d$:

1. $\int_c^d |g(y)|\mathrm{d}y$: Area between the graph of $g(y)$ and the y-axis
2. $\int_c^d |g(y)-f(y)|\mathrm{d}y$: Area between the graph of $g(y)$ and the graph of $f(y)$

✓ Volumes of revolutions about the x-axis, between $x=a$ and $x=b$:

1. $V=\pi\int_a^b (f(x))^2\mathrm{d}x$: Volume of revolution when the region between the graph of $f(x)$ and the x-axis is rotated 360° about the x-axis
2. $V=\pi\int_a^b ((f(x))^2-(g(x))^2)\mathrm{d}x$: Volume of revolution when the region between the graphs of $f(x)$ and $g(x)$ is rotated 360° about the x-axis

✓ Volumes of revolutions about the y-axis, between $y=c$ and $y=d$:

1. $V=\pi\int_c^d (g(y))^2\mathrm{d}y$: Volume of revolution when the region between the graph of $g(y)$ and the y-axis is rotated 360° about the y-axis
2. $V=\pi\int_c^d ((g(y))^2-(f(y))^2)\mathrm{d}y$: Volume of revolution when the region between the graphs of $g(y)$ and $f(y)$ is rotated 360° about the y-axis

✓ Applications in kinematics:

1. $a(t)$: Acceleration with respect to the time t
2. $v(t)=\int a(t)\mathrm{d}t$: Velocity
3. $s(t)=\int v(t)\mathrm{d}t$: Displacement
4. $d=\int_{t_1}^{t_2} |v(t)|\mathrm{d}t$: Total distance travelled between t_1 and t_2

21 Differential Equations

✓ $\frac{\mathrm{d}y}{\mathrm{d}x}=f(x,y)$: First order differential equation

✓ Solving $\frac{\mathrm{d}y}{\mathrm{d}x}=f(x)g(y)$ by separating variables:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f(x)g(y)\Rightarrow\int\frac{1}{g(y)}\mathrm{d}y=\int f(x)\mathrm{d}x$$

✓ Coupled differential equations:

1. $\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$ can be expressed as $\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

2. λ_1, λ_2: Eigenvalues of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

3. $\mathbf{v}_1$, $\mathbf{v}_2$: Eigenvectors corresponding to λ_1 and λ_2 respectively

4. $\mathbf{x} = Ae^{\lambda_1 t}\mathbf{v}_1 + Be^{\lambda_2 t}\mathbf{v}_2$: Solution of the system

5. Stable equilibrium if $\lambda_1, \lambda_2 < 0$ or $\lambda_1 = a + b\mathrm{i}$, $\lambda_2 = a - b\mathrm{i}$ and $a < 0$

6. Unstable equilibrium if $\lambda_1, \lambda_2 > 0$ or $\lambda_1 = a + b\mathrm{i}$, $\lambda_2 = a - b\mathrm{i}$ and $a > 0$

7. Saddle point if $\lambda_1 \lambda_2 < 0$

✓ Solving $\dfrac{dy}{dx} = f(x, y)$ by Euler's method, with (x_0, y_0) and step length h:

$$\begin{cases} x_{n+1} = x_n + h \\ y_{n+1} = y_n + h \left.\dfrac{dy}{dx}\right|_{(x_n,\, y_n)} \end{cases}$$

✓ Solving $\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$ by Euler's method, with (t_0, x_0, y_0) and step length h:

$t_{n+1} = t_n + h$ and $\begin{cases} x_{n+1} = x_n + h \left.\dfrac{dx}{dt}\right|_{(t_n,\, x_n,\, y_n)} \\ y_{n+1} = y_n + h \left.\dfrac{dy}{dt}\right|_{(t_n,\, x_n,\, y_n)} \end{cases}$

✓ Predator-prey models:

$\begin{cases} \dfrac{dx}{dt} = (a - by)x \\ \dfrac{dy}{dt} = (cx - d)y \end{cases}$, where a, b, c and d are positive constants

✓ The second-order differential equation $\frac{d^2x}{dt^2}+a\frac{dx}{dt}+bx=0$ can be expressed as

$$\begin{cases}\frac{dv}{dt}=-av-bx\\ \frac{dx}{dt}=v\end{cases}$$

22 Statistics

✓ Relationship between frequencies and cumulative frequencies:

Data	Frequency	Data less than or equal to	Cumulative frequency
10	f_1	10	f_1
20	f_2	20	f_1+f_2
30	f_3	30	$f_1+f_2+f_3$

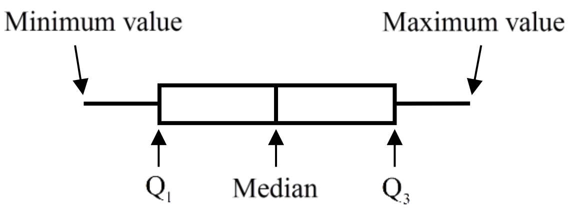
✓ Measures of central tendency for a data set $\{x_1, x_2, x_3, \cdots, x_n\}$ arranged in ascending order:

1. $\overline{x}=\frac{x_1+x_2+x_3+\cdots+x_n}{n}$: Mean
2. The datum or the average value of two data at the middle: Median
3. The datum appears the most: Mode

✓ Measures of dispersion for a data set $\{x_1, x_2, x_3, \cdots, x_n\}$ arranged in ascending order:

1. x_n-x_1: Range
2. Two subgroups A and B can be formed from the data set such that all data of the subgroup A are less than or equal to the median, while all data of the subgroup B are greater than or equal to the median
3. $Q_1=$ The median of the subgroup A: Lower quartile
4. $Q_3=$ The median of the subgroup B: Upper quartile
5. Q_3-Q_1: Inter-quartile range (IQR)
6. $\sigma=\sqrt{\frac{(x_1-\overline{x})^2+(x_2-\overline{x})^2+(x_3-\overline{x})^2+\cdots+(x_n-\overline{x})^2}{n}}$: Standard deviation

✓ Box-and-whisker diagram:



✓ A datum x is defined to be an outlier if $x < Q_1 - 1.5\text{IQR}$ or $x > Q_3 + 1.5\text{IQR}$

✓ Coding of data:

1. Only the mean, the median, the mode and the quartiles will change when each datum of the data set is added or subtracted by a value
2. All measures of central tendency and measures of dispersion will change when each datum of the data set is multiplied or divided by a value

23 Probability

✓ Terminologies:

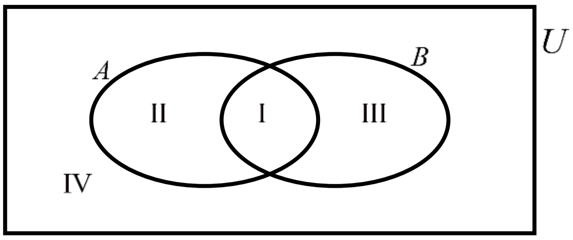
1. U : Universal set
2. A : Event
3. x : Outcome of an event
4. $n(U)$: Total number of elements
5. $n(A)$: Number of elements in A

✓ Formulae for probability:

1. $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$
2. $\text{P}(A') = 1 - \text{P}(A)$
3. $\text{P}(A \mid B) = \dfrac{\text{P}(A \cap B)}{\text{P}(B)}$
4. $\text{P}(A) = \text{P}(A \cap B) + \text{P}(A \cap B')$
5. $\text{P}(A' \cap B') + \text{P}(A \cup B) = 1$
6. $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B)$ and $\text{P}(A \cap B) = 0$ if A and B are mutually exclusive
7. $\text{P}(A \cap B) = \text{P}(A) \cdot \text{P}(B)$ and $\text{P}(A \mid B) = \text{P}(A)$ if A and B are independent

✓ Venn diagram:

1. Region I: $A \cap B$
2. Region II: $A \cap B'$
3. Region III: $A' \cap B$
4. Region IV: $(A \cup B)'$



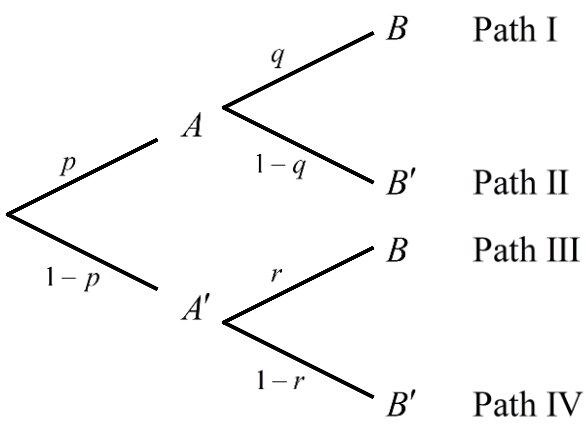
✓ Tree diagram:

1. Path I: $\mathrm{P}(A \cap B) = pq$
2. Path I + Path III:

   $= \mathrm{P}(B)$

   $= \mathrm{P}(A \cap B) + \mathrm{P}(A' \cap B)$

   $= pq + (1-p)r$



✓ Markov Chain with transition matrix $\mathbf{T}$:

1. $\det(\mathbf{T} - \lambda\mathbf{I})$: Characteristic polynomial of $\mathbf{T}$
2. Solution(s) of $\det(\mathbf{T} - \lambda\mathbf{I}) = 0$: Eigenvalue(s) of $\mathbf{T}$
3. $\mathbf{v}$: Steady state probability vector, which is the eigenvector of $\mathbf{T}$ corresponding to the eigenvalue $\lambda = 1$
4. $\mathbf{v}_0$: Initial state probability vector
5. $\mathbf{v}_n = \mathbf{T}^n\mathbf{v}_0$: State probability vector after n transitions
6. The column sum of $\mathbf{T}$ must be equal to 1

24 Discrete Probability Distributions

✓ Properties of a discrete random variable X:

X	x_1	x_2	...	x_n
$\mathrm{P}(X = x)$	$\mathrm{P}(X = x_1)$	$\mathrm{P}(X = x_2)$	...	$\mathrm{P}(X = x_n)$

1. $\mathrm{P}(X = x_1) + \mathrm{P}(X = x_2) + \cdots + \mathrm{P}(X = x_n) = 1$
2. $\mathrm{E}(X) = x_1\mathrm{P}(X = x_1) + x_2\mathrm{P}(X = x_2) + \cdots + x_n\mathrm{P}(X = x_n)$: Expected value of X
3. $\mathrm{E}(X) = 0$ if a fair game is considered

25 Binomial Distribution

✓ Properties of a random variable $X \sim \mathrm{B}(n, p)$ following binomial distribution:

1. Only two outcomes from every independent trial (Success and failure)
2. n : Number of trials
3. p : Probability of success
4. X : Number of successes in n trials

✓ Formulae for binomial distribution:

1. $\mathrm{P}(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$ for $0 \le r \le n$, $r \in \mathbb{Z}$
2. $\mathrm{E}(X) = np$: Expected value of X
3. $\mathrm{Var}(X) = np(1-p)$: Variance of X
4. $\sqrt{np(1-p)}$: Standard deviation of X
5. $\mathrm{P}(X \le r) = \mathrm{P}(X < r+1) = 1 - \mathrm{P}(X \ge r+1)$

26 Poisson Distribution

✓ Properties of a random variable $X \sim \mathrm{Po}(\lambda)$ following Poisson distribution:

1. The expected number of occurrences of an event is directly proportional to the length of the time interval
2. The numbers of occurrences of the event in different disjoint time intervals are independent
3. λ : Mean number of occurrences of an event
4. X : Number of occurrences of an event

✓ Formulae for Poisson distribution:

1. $\mathrm{P}(X = r) = \dfrac{e^{-\lambda} \cdot \lambda^r}{r!}$ for $r \ge 0$, $r \in \mathbb{Z}$
2. $\mathrm{E}(X) = \lambda$: Expected value of X
3. $\mathrm{Var}(X) = \lambda$: Variance of X
4. $\sqrt{\lambda}$: Standard deviation of X
5. $\mathrm{P}(X \le r) = \mathrm{P}(X < r+1) = 1 - \mathrm{P}(X \ge r+1)$

27 Normal Distribution

✓ Properties of a random variable $X \sim \mathrm{N}(\mu, \sigma^2)$ following normal distribution:

1. μ: Mean
2. σ: Standard deviation
3. The mean, the median and the mode are the same
4. The normal curve representing the distribution is a bell-shaped curve which is symmetric about the middle vertical line
5. $\mathrm{P}(X < \mu) = \mathrm{P}(X > \mu) = 0.5$
6. The total area under the curve is 1

28 Linear Combinations of Variables

✓ Properties of linear transformations of independent random variables:

1. $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$
2. $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$
3. $\mathrm{E}(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1\mathrm{E}(X_1) + a_2\mathrm{E}(X_2) + \cdots + a_n\mathrm{E}(X_n)$
4. $\mathrm{Var}(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1^2\mathrm{Var}(X_1) + a_2^2\mathrm{Var}(X_2) + \cdots + a_n^2\mathrm{Var}(X_n)$

29 Point Estimation

✓ Central limit theorem:

1. $X_i \sim N(\mu, \sigma^2)$
2. $\bar{X} = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$: Sample mean
3. $\bar{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$ when n is sufficiently large

✓ Properties of point estimation:

1. $s_{n-1}^2 = \dfrac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$: Sample variance

2. $s_n^2 = \dfrac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}$

3. $s_{n-1}^2 = \dfrac{n}{n-1} s_n^2$

4. $\mathrm{E}(\bar{X}) = X$

5. $\mathrm{E}(s_{n-1}^2) = \sigma^2$

30 Interval Estimation

✓ $(1-\alpha)\%$ confidence interval for population mean μ when the population variance σ^2 is known:

1. $\bar{X} = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$: Sample mean

2. $\left(\bar{X} - z_{\frac{\alpha}{2}} \cdot \dfrac{\sigma}{\sqrt{n}}, \bar{X} + z_{\frac{\alpha}{2}} \cdot \dfrac{\sigma}{\sqrt{n}}\right)$, where $P\left(Z > z_{\frac{\alpha}{2}}\right) = \dfrac{\alpha}{2}$

3. $2z_{\frac{\alpha}{2}} \cdot \dfrac{\sigma}{\sqrt{n}}$: Interval width

✓ $(1-\alpha)\%$ confidence interval for population mean μ when the population variance σ^2 is unknown:

1. $\bar{X} = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$: Sample mean

2. $\left(\bar{X} - t_{n-1,\frac{\alpha}{2}} \cdot \dfrac{s_{n-1}}{\sqrt{n}}, \bar{X} + t_{n-1,\frac{\alpha}{2}} \cdot \dfrac{s_{n-1}}{\sqrt{n}}\right)$, where $P\left(\bar{X} > t_{n-1,\frac{\alpha}{2}}\right) = \dfrac{\alpha}{2}$

3. $2t_{n-1,\frac{\alpha}{2}} \cdot \dfrac{s_{n-1}}{\sqrt{n}}$: Interval width

31 Bivariate Analysis

✓ Correlations:

Positive	Strong	$0.75 < r < 1$
	Moderate	$0.5 < r < 0.75$
	Weak	$0 < r < 0.5$
No		$r = 0$
Negative	Weak	$-0.5 < r < 0$
	Moderate	$-0.75 < r < -0.5$
	Strong	$-1 < r < -0.75$

where r is the correlation coefficient

✓ Linear regression:

$y = ax + b$: Regression line of y on x

✓ Correlation Coefficient for ranked data:

r_s: Spearman's Rank Correlation Coefficient

✓ Coefficient of determination:

R^2: Coefficient of determination

R^2 % of the variability of the data can be explained by the regression model

✓ Sum of square residuals:

x	x_1	x_2	$\cdots$	x_n
y	y_1	y_2	$\cdots$	y_n
Predicted value of y	$\hat{y}_1$	$\hat{y}_2$	$\cdots$	$\hat{y}_n$

$SS_{res} = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$: Sum of square residuals

✓ Non-linear regressions:

1. Quadratic regression
2. Cubic regression
3. Quartic regression
4. Exponential regression
5. Logarithmic regression
6. Power regression
7. Logistic regression

32 Statistical Tests

✓ Hypothesis test:

H_0: Null hypothesis

H_1: Alternative hypothesis

C: Critical value in the hypothesis test

α: Significance level

✓ χ^2 test for independence for a contingency table with r rows and c columns:

$n = rc$: Total number of data

O_i ($i = 1, 2, \ldots, n$): Observed frequencies

E_i ($i = 1, 2, \ldots, n$): Expected frequencies

$v = (r-1)(c-1)$: Degree of freedom

$\chi^2_{calc} = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$: χ^2 test statistic

H_0: Two variables are independent

H_1: Two variables are not independent

H_0 is rejected if $\chi^2_{calc} > C$ or the p-value is less than the significance level

H_0 is not rejected if $\chi^2_{calc} < C$ or the p-value is greater than the significance level

✓ χ^2 goodness of fit test for a contingency table with 1 row and c columns:

$v = c - 1$: Degree of freedom

H_0: The data follows an assigned distribution

H_1: The data does not follow an assigned distribution

H_0 is rejected if $\chi^2_{calc} > C$ or the p-value is less than the significance level

H_0 is not rejected if $\chi^2_{calc} < C$ or the p-value is greater than the significance level

✓ Two sample t test:

μ_1, μ_2: The population means of two groups of data

H_0: $\mu_1 = \mu_2$

H_1: $\mu_1 > \mu_2$, $\mu_1 < \mu_2$ (for 1-tailed test), $\mu_1 \neq \mu_2$ (for 2-tailed test)

H_0 is rejected if the p-value is less than the significance level

H_0 is not rejected if the p-value is greater than the significance level

✓ More about χ^2 test for independence and χ^2 goodness of fit test:
All expected frequencies E_i should be greater than 5

✓ Z test when the population variance σ^2 is known:
μ: Population mean
H_0: $\mu = \mu_0$
H_1: $\mu > \mu_0$, $\mu < \mu_0$ (for 1-tailed test), $\mu \neq \mu_0$ (for 2-tailed test)
H_0 is rejected if the p-value is less than the significance level
H_0 is not rejected if the p-value is greater than the significance level

✓ One sample t test when the population variance σ^2 is unknown:
μ: Population mean
H_0: $\mu = \mu_0$
H_1: $\mu > \mu_0$, $\mu < \mu_0$ (for 1-tailed test), $\mu \neq \mu_0$ (for 2-tailed test)
H_0 is rejected if the p-value is less than the significance level
H_0 is not rejected if the p-value is greater than the significance level

✓ Paired t test:
$d = x - y$: Difference between each pair of data from two variables x and y
H_0: $\mu_d = 0$
H_1: $\mu_d > 0$, $\mu_d < 0$ (for 1-tailed test), $\mu_d \neq 0$ (for 2-tailed test)
H_0 is rejected if the p-value is less than the significance level
H_0 is not rejected if the p-value is greater than the significance level

✓ Test involving binomial distribution:
$X \sim \mathrm{B}(n, p)$
H_0: $p = p_0$
H_1: $p > p_0$, $p < p_0$
x: Observed value of X
$\mathrm{P}(X \geq x)$ for H_1: $p > p_0$ or $\mathrm{P}(X \leq x)$ for H_1: $p < p_0$: p-value under $X \sim \mathrm{B}(n, p_0)$
H_0 is rejected if the p-value is less than the significance level
H_0 is not rejected if the p-value is greater than the significance level

✓ Test involving Poisson distribution:
$X \sim \text{Po}(\lambda)$
$H_0:\ \lambda = \lambda_0$
$H_1:\ \lambda > \lambda_0,\ \lambda < \lambda_0$
x: Observed value of X
$\text{P}(X \geq x)$ for $H_1:\ \lambda > \lambda_0$ or $\text{P}(X \leq x)$ for $H_1:\ \lambda < \lambda_0$: p-value under $X \sim \text{Po}(\lambda_0)$
H_0 is rejected if the p-value is less than the significance level
H_0 is not rejected if the p-value is greater than the significance level

✓ Test involving bivariate normal distribution:
ρ: Product moment correlation coefficient
$H_0:\ \rho = 0$
$H_1:\ \rho > 0,\ \rho < 0$ (for 1-tailed test), $\rho \neq 0$ (for 2-tailed test)
H_0 is rejected if the p-value is less than the significance level
H_0 is not rejected if the p-value is greater than the significance level

✓ Type I and type II errors:
α: Significance level
$\text{P(Reject } H_0 \mid H_0 \text{ is true)} = \alpha$: Type I error
$\text{P(Not reject } H_0 \mid H_0 \text{ is not true)}$: Type II error

33 Paper 3 Analysis

✓ Nature of paper: Structured question

✓ Time allowed: 60 minutes

✓ Maximum mark: 55 marks

✓ Number of questions: 2

✓ Mark range per question: 25 marks to 30 marks

✓ Weighting: 20% of the total mark

✓ Ways of assessing:

1. Identify a hypothesis test
2. State
   (a) a condition
   (b) a reason
   (c) an assumption
3. Write down
   (a) the value of a quantity
   (b) the formula of a quantity
4. Find
   (a) the value of a quantity
   (b) the formula of a quantity
5. Solve an equation
6. Perform a hypothesis test
7. Show
   (a) a quantity equals to a value
   (b) the formula of a quantity
8. Estimate the value of a quantity
9. Predict the value of a quantity
10. Sketch a graph
11. Suggest an improvement
12. Explain a reason
13. Describe a result
14. Verify
    (a) the value of a quantity
    (b) the expression of a quantity
15. Comment on
    (a) the validity of an argument
    (b) a statement

Notes