Chapter 9 Solution

Exercise 33

1. (a) $(\sqrt{3}+i)^3$

$= (\sqrt{3})^3 + \binom{3}{1}(\sqrt{3})^2 i + \binom{3}{2}(\sqrt{3})i^2 + i^3$ M1A1

$= 3\sqrt{3} + (3)(3)i + (3)(\sqrt{3})(-1) + (-i)$ M1A1

$= 8i$ A1

[5]

(b) -8 A1

[1]

2. $\frac{z}{1-z} = -1-0.5i$

$z = (-1-0.5i)(1-z)$ M1

$z = -1 + z - 0.5i + 0.5iz$

$1 + 0.5i = 0.5iz$

$z = \frac{1+0.5i}{0.5i}$ A1

$z = \frac{(1+0.5i)(-i)}{0.5i(-i)}$ M1

$z = \frac{-i-0.5i^2}{-0.5i^2}$

$z = \frac{-i+0.5}{0.5}$ A1

$z = 1-2i$

Thus, the imaginary part of z is -2. A1

[5]

3. Let $z = a + b\text{i}$.

$z - |z| = -4(2 - \text{i})$

$a + b\text{i} - \sqrt{a^2 + b^2} = -8 + 4\text{i}$ M1A1

$(a - \sqrt{a^2 + b^2}) + b\text{i} = -8 + 4\text{i}$

$b = 4$ A1

$a - \sqrt{a^2 + 4^2} = -8$

$a + 8 = \sqrt{a^2 + 16}$

$(a + 8)^2 = a^2 + 16$ M1

$a^2 + 16a + 64 = a^2 + 16$

$16a = -48$

$a = -3$

Thus, the real part of z is -3. A1

[5]

4. Let $z = a + b\text{i}$.

$|z| - 18 = (z - 12)\text{i}$

$\sqrt{a^2 + b^2} - 18 = (a + b\text{i} - 12)\text{i}$ M1A1

$\sqrt{a^2 + b^2} - 18 = a\text{i} + b\text{i}^2 - 12\text{i}$

$\sqrt{a^2 + b^2} - 18 = -b + (a - 12)\text{i}$ A1

$a - 12 = 0$

$a = 12$ A1

$\sqrt{12^2 + b^2} - 18 = -b$

$\sqrt{144 + b^2} = 18 - b$

$144 + b^2 = (18 - b)^2$ M1

$144 + b^2 = 324 - 36b + b^2$

$-180 = -36b$

$b = 5$

Thus, the imaginary part of z is 5. A1

[6]

Exercise 34

1. (a) $z_C = 3 - 3i + 2(12 + 5i)$ M1

$z_C = 27 + 7i$

$z_D = -2 + 9i + 2(12 + 5i)$ M1

$z_D = 22 + 19i$

Thus, the complex numbers represented by the points C and D are $27 + 7i$ and $22 + 19i$ respectively. A2

[4]

(b) The area of ABCD

$= (AB)(AD)$ M1

$= 2AB^2$

$= 2(\sqrt{5^2 + 12^2})^2$

$= 338$ A1

[2]

2. (a) $z_B = -18 + 10i + 20$ M1

$z_B = 2 + 10i$

$z_C = -18 + 10i + (10 - (20\sin 60^\circ)i)$ M1

$z_C = -8 + (10 - 10\sqrt{3})i$

Thus, the complex numbers represented by the points B and C are $2 + 10i$ and $-8 + (10 - 10\sqrt{3})i$ respectively. A2

[4]

(b) The area of ABC

$= \dfrac{(20)(20\sin 60^\circ)}{2}$ M1

$= \dfrac{(20)(10\sqrt{3})}{2}$

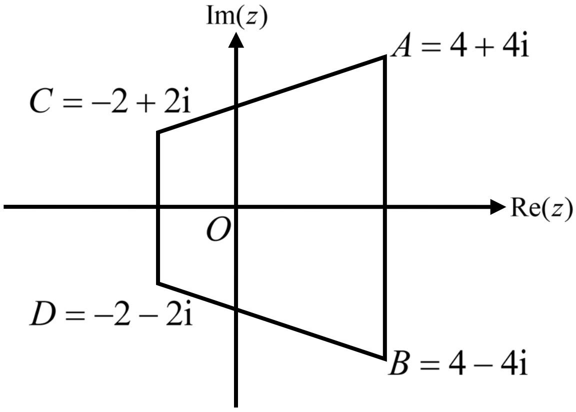
$= 100\sqrt{3}$ A1

[2]

3. (a) For any two correct points A1

For all correct points A1

For sketching a trapezium A1



[3]

(b) $\arg(\omega) = \arctan\left(\dfrac{2}{-2}\right)$ M1

$\arg(\omega) = \dfrac{3\pi}{4}$ A1

[2]

(c) The area of the quadrilateral ABDC

$= \dfrac{(4+8)(6)}{2}$ M1

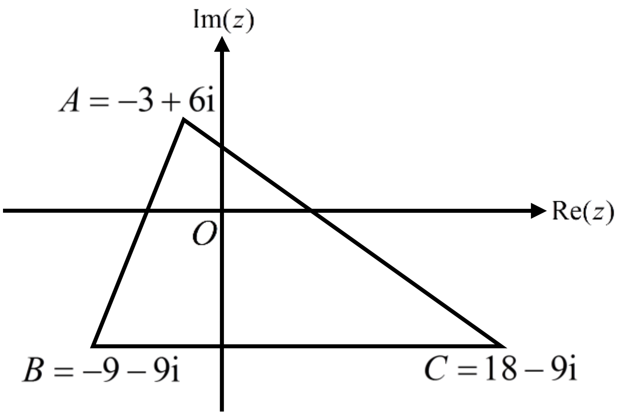
$= 36$ A1

[2]

4. (a) For any two correct points A1

For all correct points A1

For sketching a triangle A1



[3]

(b) $\arg(3z-27\mathrm{i})=\arctan\left(\frac{-9}{-9}\right)$ M1

$\arg(\omega)=\frac{5\pi}{4}$ A1

[2]

(c) The area of the triangle ABC

$=\frac{(27)(15)}{2}$ M1

$=202.5$ A1

[2]

Exercise 35

1. (a) $z_3 = \sqrt{3} + \mathrm{i}$ A1

[1]

(b) The cubic polynomial

$= (z-(-2))(z-(\sqrt{3}+\mathrm{i}))(z-(\sqrt{3}-\mathrm{i}))$ M1

$= (z+2)(z^2 - 2\sqrt{3}z + 4)$

$= z^3 + (2-2\sqrt{3})z^2 + (4-4\sqrt{3})z + 8$

$= z^3 + 2(1-\sqrt{3})z^2 + 4(1-\sqrt{3})z + 8$

$\therefore b = 2$, $c = 4$ and $d = 8$ A3

[4]

(c) $z_3 = \sqrt{3} + \mathrm{i}$

$z_3 = (\sqrt{(\sqrt{3})^2 + 1^2})e^{\mathrm{i}\arctan\left(\frac{1}{\sqrt{3}}\right)}$

$z_3 = 2\mathrm{e}^{\frac{\pi}{6}\mathrm{i}}$ A2

[2]

2. (a) A polynomial function $f(x)$ of degree 4 has four roots. R1

There are two real roots only as the graph of $f(x)$ has only two x-intercepts. R1

Thus, there are two complex roots for the equation $f(x) = 0$. AG

[2]

(b) The another complex root is $3 + \mathrm{i}$. A1

$f(x) = (x-(-5))(x-1)(x-(3+\mathrm{i}))(x-(3-\mathrm{i}))$ M1A1

$f(x) = (x+5)(x-1)(x^2+6x+10)$ A1

[4]

3. The third root is $1-2i$. A1

The fourth root

$=3-4-(1+2i)-(1-2i)$

$=-3$ A1

The quartic polynomial

$=(z-4)(z-(-3))(z-(1+2i))(z-(1-2i))$ M1

$=(z^2-z-12)(z^2-2z+5)$ A1

$=z^4-3z^3-5z^2+19z-60$ A1

$\therefore b=-3,\ c=-5,\ d=19$ and $e=-60$

$b+c+d+e+49$

$=-3+(-5)+19+(-60)+49$ M1A1

$=0$ AG

[7]

4. The third root is $2+5i$. A1

The fourth root

$=\dfrac{174}{(-3)(2+5i)(2-5i)}$

$=-2$ A1

The quartic polynomial

$=(z-(-3))(z-(-2))(z-(2+5i))(z-(2-5i))$ M1

$=(z^2+5z+6)(z^2-4z+29)$ A1

$=z^4+z^3+15z^2+121z+174$ A1

$\therefore b=1,\ c=15,\ d=121$ and $e=174$

$\sqrt{b+c}+\sqrt{d}-15$

$=\sqrt{1+15}+\sqrt{121}-15$ M1A1

$=0$ AG

[7]

Exercise 36

1. (a) $z^9 = 1$

$z^9 = \cos 0 + \mathrm{i}\sin 0$ A1

$z = \cos\left(\frac{0+2k\pi}{9}\right) + \mathrm{i}\sin\left(\frac{0+2k\pi}{9}\right)$ M1

($k = 0, 1, 2, 3, 4, 5, 6, 7, 8$)

$z = \cos 0 + \mathrm{i}\sin 0$, $z = \cos\frac{2\pi}{9} + \mathrm{i}\sin\frac{2\pi}{9}$,

$z = \cos\frac{4\pi}{9} + \mathrm{i}\sin\frac{4\pi}{9}$, $z = \cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}$,

$z = \cos\frac{8\pi}{9} + \mathrm{i}\sin\frac{8\pi}{9}$, $z = \cos\frac{10\pi}{9} + \mathrm{i}\sin\frac{10\pi}{9}$,

$z = \cos\frac{4\pi}{3} + \mathrm{i}\sin\frac{4\pi}{3}$, $z = \cos\frac{14\pi}{9} + \mathrm{i}\sin\frac{14\pi}{9}$

or $z = \cos\frac{16\pi}{9} + \mathrm{i}\sin\frac{16\pi}{9}$ A2

[4]

(b) (i) $(z^3 - 1)(z^6 + z^3 + 1)$

$= z^9 + z^6 + z^3 - z^6 - z^3 - 1$ M1

$= z^9 - 1$ A1

(ii) $z^6 + z^3 + 1 = 0$

$\frac{z^9 - 1}{z^3 - 1} = 0$, where $z \neq 1$, $z \neq \operatorname{cis}\frac{2\pi}{3}$ and

$z \neq \operatorname{cis}\frac{4\pi}{3}$ M1

$z = \operatorname{cis}\frac{2\pi}{9}$, $z = \operatorname{cis}\frac{4\pi}{9}$, $z = \operatorname{cis}\frac{8\pi}{9}$,

$z = \operatorname{cis}\frac{10\pi}{9}$, $z = \operatorname{cis}\frac{14\pi}{9}$ or $z = \operatorname{cis}\frac{16\pi}{9}$ A1

[4]

(c) (i) $\alpha^* = (w + w^4 + w^7)^*$

$\alpha^* = (w)^* + (w^4)^* + (w^7)^*$ M1

$\alpha^* = w^{9-1} + w^{9-4} + w^{9-7}$ A1

$\alpha^* = w^2 + w^5 + w^8$ A1

(ii) $1 + \alpha + \alpha^* = -\frac{b}{1}$ M1A1

$1 + w + w^4 + w^7 + w^2 + w^5 + w^8 = -b$

$1 + w(1 + w^3 + w^6) + w^2(1 + w^3 + w^6) = -b$ M1

$1 + 0 + 0 = -b$

$b = -1$ A1

(iii) $(1)(\alpha)(\alpha^*) = -\frac{d}{1}$ M1A1

$(w + w^4 + w^7)(w^2 + w^5 + w^8) = -d$

$(w)(1 + w^3 + w^6)(w^2)(1 + w^3 + w^6) = -d$ M1

$(w)(0)(w^2)(0) = -d$

$d = 0$ A1

(iv) $1^3 - 1^2 + c(1) + 0 = 0$ M1

$c = 0$ A1

[13]

2. (a) $z^5 + 1 = 0$

$z^5 = -1$

$z^5 = \cos\pi + \mathrm{i}\sin\pi$ A1

$z = \cos\left(\frac{\pi + 2k\pi}{5}\right) + \mathrm{i}\sin\left(\frac{\pi + 2k\pi}{5}\right)$ M1

($k = 0, 1, 2, 3, 4$)

$z = \cos\frac{\pi}{5} + \mathrm{i}\sin\frac{\pi}{5}$, $z = \cos\frac{3\pi}{5} + \mathrm{i}\sin\frac{3\pi}{5}$,

$z = \cos\pi + \mathrm{i}\sin\pi$, $z = \cos\frac{7\pi}{5} + \mathrm{i}\sin\frac{7\pi}{5}$ or

$z = \cos\frac{9\pi}{5} + \mathrm{i}\sin\frac{9\pi}{5}$ A2

[4]

(b) (i) $(z+1)(z^4 - z^3 + z^2 - z + 1)$

$= z^5 - z^4 + z^3 - z^2 + z + z^4 - z^3 + z^2 - z + 1$ M1

$= z^5 + 1$ A1

(ii) $z^4 - z^3 + z^2 - z + 1 = 0$

$\frac{z^5 + 1}{z + 1} = 0$, where $z \neq -1$ M1

$z = \cos\frac{\pi}{5} + \mathrm{i}\sin\frac{\pi}{5}$, $z = \cos\frac{3\pi}{5} + \mathrm{i}\sin\frac{3\pi}{5}$,

$z = \cos\frac{7\pi}{5} + \mathrm{i}\sin\frac{7\pi}{5}$ or

$z = \cos\frac{9\pi}{5} + \mathrm{i}\sin\frac{9\pi}{5}$ A1

[4]

$$\left(\cos\frac{\pi}{5}+\mathrm{i}\sin\frac{\pi}{5}\right)+\left(\cos\frac{3\pi}{5}+\mathrm{i}\sin\frac{3\pi}{5}\right)$$

(c) (i) $$+\left(\cos\frac{7\pi}{5}+\mathrm{i}\sin\frac{7\pi}{5}\right)+\left(\cos\frac{9\pi}{5}+\mathrm{i}\sin\frac{9\pi}{5}\right)$$ M1A1

$$=-\frac{-1}{1}$$

$$\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}+\cos\frac{7\pi}{5}+\cos\frac{9\pi}{5}$$

$$+\mathrm{i}\left(\sin\frac{\pi}{5}+\sin\frac{3\pi}{5}+\sin\frac{7\pi}{5}+\sin\frac{9\pi}{5}\right)$$

$$=1$$

$$\therefore\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}+\cos\frac{7\pi}{5}+\cos\frac{9\pi}{5}=1$$ M1

$$\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}+\cos\left(2\pi-\frac{3\pi}{5}\right)$$

$$+\cos\left(2\pi-\frac{\pi}{5}\right)=1$$

$$2\cos\frac{\pi}{5}+2\cos\frac{3\pi}{5}=1$$ M1

$$\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$$ A1

(ii) $$\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{5}\right)+\sin\left(\frac{\pi}{2}-\frac{3\pi}{5}\right)=\frac{1}{2}$$ M1

$$\sin\frac{3\pi}{10}+\sin\left(-\frac{\pi}{10}\right)=\frac{1}{2}$$

$$\sin\frac{3\pi}{10}-\sin\frac{\pi}{10}=\frac{1}{2}$$ M1

$$\therefore\sin\frac{\pi}{10}-\sin\frac{3\pi}{10}=-\frac{1}{2}$$ A1

(iii) $$\sum_{r=1}^{5}\cos\frac{(2r-1)\pi}{5}$$

$$=\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}+\cos\pi+\cos\frac{7\pi}{5}+\cos\frac{9\pi}{5}$$

$$=\left(\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}+\cos\frac{7\pi}{5}+\cos\frac{9\pi}{5}\right)$$

$+\cos\pi$

$=1+(-1)$ M1

$=0$ A1

[10]

3. (a) (i) $z^3 - 1 = 0$

$z^3 = 1$

$z^3 = \cos 0 + \mathrm{i}\sin 0$ A1

$z = \cos\left(\frac{0+2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{0+2k\pi}{3}\right)$ M1

$(k = 0, 1, 2)$

$\therefore w = \cos\left(\frac{0+2(1)\pi}{3}\right) + \mathrm{i}\sin\left(\frac{0+2(1)\pi}{3}\right)$

$w = \cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}$ A1

(ii) $w^3 - 1 = 0$

$(w-1)(w^2 + w + 1) = 0$ M1

$w^2 + w + 1 = 0$ A1

$\therefore w^3 + w^2 + w = 0$ AG

[5]

(b) (i) $\beta = \alpha^*$

$\beta = (w^3 + w^4)^*$ M1

$\beta = (1 + w)^*$

$\beta = 1 + w^*$

$\beta = 1 + w^{3-1}$ A1

$\beta = 1 + w^2$ A1

(ii) $1 + (-1) + \alpha + \beta = -\frac{b}{1}$ M1A1

$1 + w + 1 + w^2 = -b$

$1 + 0 = -b$

$b = -1$ A1

(iii) $(1)(-1)(\alpha)(\beta) = \frac{e}{1}$ M1A1

$-(1 + w)(1 + w^2) = e$

$-(1 + w + w^2 + w^3) = e$

$-(0 + 1) = e$

$e = -1$ A1

(iv)	$(1)(-1)+(1)(\alpha)+(1)(\beta)$ $+(-1)(\alpha)+(-1)(\beta)+(\alpha)(\beta)=\frac{c}{1}$	M1
	$-1+\alpha\beta=c$	A1
	$-1+(1+w)(1+w^2)=c$	
	$-1+1+w+w^2+w^3=c$	A1
	$c=0$	AG
	$1^4-1^3+d(1)-1=0$	M1A1
	$d=1$	AG

[14]

4. (a) $z^7 + 1 = 0$

$z^7 = -1$

$z^7 = \cos\pi + i\sin\pi$ A1

$z = \cos\left(\frac{\pi + 2k\pi}{7}\right) + i\sin\left(\frac{\pi + 2k\pi}{7}\right)$ M1

($k = 0, 1, 2, 3, 4, 5, 6$)

$z = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$, $z = \cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}$,

$z = \cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}$, $z = \cos\pi + i\sin\pi$,

$z = \cos\frac{9\pi}{7} + i\sin\frac{9\pi}{7}$, $z = \cos\frac{11\pi}{7} + i\sin\frac{11\pi}{7}$ or

$z = \cos\frac{13\pi}{7} + i\sin\frac{13\pi}{7}$ A2

[4]

(b) $z^7 + 1$

$= z^7 - z^6 + z^5 - z^4 + z^3 - z^2 + z$

$+ z^6 - z^5 + z^4 - z^3 + z^2 - z + 1$ M1

$= z(z^6 - z^5 + z^4 - z^3 + z^2 - z + 1)$

$+ z^6 - z^5 + z^4 - z^3 + z^2 - z + 1$

$= (z+1)(z^6 - z^5 + z^4 - z^3 + z^2 - z + 1)$ A1

$z^6 - z^5 + z^4 - z^3 + z^2 - z + 1 = 0$

$\frac{z^7 + 1}{z + 1} = 0$, where $z \neq -1$ M1

$z = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$, $z = \cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}$,

$z = \cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}$, $z = \cos\frac{9\pi}{7} + i\sin\frac{9\pi}{7}$,

$z = \cos\frac{11\pi}{7} + i\sin\frac{11\pi}{7}$ or $z = \cos\frac{13\pi}{7} + i\sin\frac{13\pi}{7}$ A1

[4]

(c)	(i)	$(z-p)(z-q)=0$	M1
		$z^2-(p+q)z+pq=0$	
		$p+q=\lambda^4+\lambda^2-\lambda+\lambda^6-\lambda^5-\lambda^3$	M1
		$p+q=\lambda^6-\lambda^5+\lambda^4-\lambda^3+\lambda^2-\lambda$	
		$p+q=-1$	A1
		$pq=(\lambda^4+\lambda^2-\lambda)(\lambda^6-\lambda^5-\lambda^3)$	M1
		$pq=\lambda^{10}-\lambda^9-\lambda^7+\lambda^8-\lambda^7-\lambda^5$ $-\lambda^7+\lambda^6+\lambda^4$	M1
		$pq=\lambda^7(\lambda^3)-\lambda^7(\lambda^2)-\lambda^7$ $+\lambda^7(\lambda)-\lambda^7-\lambda^5-\lambda^7+\lambda^6+\lambda^4$	
		$pq=-\lambda^3+\lambda^2+1-\lambda+1-\lambda^5+1+\lambda^6+\lambda^4$	M1
		$pq=\lambda^6-\lambda^5+\lambda^4-\lambda^3+\lambda^2-\lambda+1+2$	
		$pq=0+2$	
		$pq=2$	A1
		$\therefore z^2+z+2=0$	A1
	(ii)	$(z-(p+1))(z-(q+1))=0$	M1
		$z^2-(p+1+q+1)z+(p+1)(q+1)=0$	
		$z^2-(p+q+2)z+(pq+p+q+1)=0$	A1
		$z^2-(-1+2)z+(2-1+1)=0$	M1
		$z^2-z+2=0$	A1

[12]

Exercise 37

1. (a) $z = \dfrac{2+\mathrm{i}}{2-\mathrm{i}}$

$z = \dfrac{(2+\mathrm{i})(2+\mathrm{i})}{(2-\mathrm{i})(2+\mathrm{i})}$ M1

$z = \dfrac{4+4\mathrm{i}+\mathrm{i}^2}{5}$

$z = \dfrac{4+4\mathrm{i}+(-1)}{5}$

$z = \dfrac{3}{5}+\dfrac{4}{5}\mathrm{i}$ A1

[2]

(b) The modulus of z

$= \sqrt{\left(\dfrac{3}{5}\right)^2+\left(\dfrac{4}{5}\right)^2}$ M1

$= \sqrt{1}$

$= 1$ A1

[2]

(c) The argument of z

$= \arctan\left(\dfrac{\frac{4}{5}}{\frac{3}{5}}\right)$ M1

$= \arctan\left(\dfrac{4}{3}\right)$

$= 0.927295218$

$= 0.927 \text{ rad}$ A1

[2]

2. $\omega^* = (1+\cos\theta) + \mathrm{i}\sin\theta$ A1

$(\omega^*)^2 = ((1+\cos\theta) + \mathrm{i}\sin\theta)^2$ M1

$(\omega^*)^2 = (1+\cos\theta)^2 + 2\mathrm{i}\sin\theta(1+\cos\theta) + \mathrm{i}^2\sin^2\theta$

$(\omega^*)^2 = 1 + 2\cos\theta + \cos^2\theta + 2\mathrm{i}\sin\theta(1+\cos\theta) - \sin^2\theta$

$(\omega^*)^2 = 2\cos\theta + 2\cos^2\theta + 2\mathrm{i}\sin\theta(1+\cos\theta)$ (A1) for simplification

$(\omega^*)^2 = 2\cos\theta(1+\cos\theta) + 2\mathrm{i}\sin\theta(1+\cos\theta)$ A1

The modulus of $(\omega^*)^2$

$= \sqrt{(2\cos\theta(1+\cos\theta))^2 + (2\sin\theta(1+\cos\theta))^2}$ M1

$= \sqrt{4\cos^2\theta(1+\cos\theta)^2 + 4\sin^2\theta(1+\cos\theta)^2}$

$= \sqrt{4(1+\cos\theta)^2}$

$= 2|1+\cos\theta|$ A1

The argument of $(\omega^*)^2$

$= \arctan\left(\dfrac{2\sin\theta(1+\cos\theta)}{2\cos\theta(1+\cos\theta)}\right)$ M1

$= \arctan(\tan\theta)$

$= \theta$ A1

[8]

3. $1+\omega=(1+\sin\theta)+\mathrm{i}\cos\theta$

$(1+\omega)^2=((1+\sin\theta)+\mathrm{i}\cos\theta)^2$ M1A1

$(1+\omega)^2=(1+\sin\theta)^2+2\mathrm{i}\cos\theta(1+\sin\theta)+\mathrm{i}^2\cos^2\theta$

$(1+\omega)^2=1+2\sin\theta+\sin^2\theta+2\mathrm{i}\cos\theta(1+\sin\theta)-\cos^2\theta$

$(1+\omega)^2=2\sin\theta+2\sin^2\theta+2\mathrm{i}\cos\theta(1+\sin\theta)$ (A1) for simplification

$(1+\omega)^2=2\sin\theta(1+\sin\theta)+2\mathrm{i}\cos\theta(1+\sin\theta)$ A1

The modulus of $(1+\omega)^2$

$=\sqrt{(2\sin\theta(1+\sin\theta))^2+(2\cos\theta(1+\sin\theta))^2}$ M1

$=\sqrt{4\sin^2\theta(1+\sin\theta)^2+4\cos^2\theta(1+\sin\theta)^2}$

$=\sqrt{4(1+\sin\theta)^2}$

$=2|1+\sin\theta|$ A1

The argument of $(1+\omega)^2$

$=\arctan\left(\dfrac{2\cos\theta(1+\sin\theta)}{2\sin\theta(1+\sin\theta)}\right)$ M1

$=\arctan(\cot\theta)$

$=\arctan\left(\tan\left(\dfrac{\pi}{2}-\theta\right)\right)$

$=\dfrac{\pi}{2}-\theta$ A1

$\therefore(1+\omega)^2=2|1+\sin\theta|e^{\left(\frac{\pi}{2}-\theta\right)\mathrm{i}}$ A1

[9]

4. (a) $z_1 + iz_2 = 0$

$z_1 = -iz_2$

$(-iz_2)z_2 = -2\sqrt{a} + 2i$ M1

$-iz_2^{\ 2} = -2\sqrt{a} + 2i$

$z_2^{\ 2} = \dfrac{2\sqrt{a}}{i} - 2$ A1

$z_2^{\ 2} = -2 - 2\sqrt{a}i$

$\left|z_2^{\ 2}\right| = \left|-2 - 2\sqrt{a}i\right|$

$\left|z_2\right|^2 = \sqrt{(-2)^2 + (-2\sqrt{a})^2}$ M1

$2^2 = \sqrt{4 + 4a}$

$16 = 4 + 4a$

$12 = 4a$

$a = 3$ A1

[4]

(b) $|z_1| = |-iz_2|$

$|z_1| = |-i||z_2|$ M1

$|z_1| = (1)(2)$

$|z_1| = 2$ A1

[2]

(c) $z_2^{\ 2} = -2 - 2\sqrt{3}i$

$\arg(z_2^{\ 2}) = \arctan\left(\dfrac{-2\sqrt{3}}{-2}\right)$ M1

$2\arg(z_2) = \arctan(\sqrt{3})$

$2\arg(z_2) = \dfrac{\pi}{3}$

$\arg(z_2) = \dfrac{\pi}{6}$ A1

$\therefore z_2 = 2e^{\frac{\pi}{6}i}$ A1

[3]

Exercise 38

1.	(a)	$(-\mathrm{i})^4-8(-\mathrm{i})^3+13(-\mathrm{i})^2-8(-\mathrm{i})+12$	M1	
		$=1-8\mathrm{i}+13(-1)-8(-\mathrm{i})+12$	A1	
		$=1-8\mathrm{i}-13+8\mathrm{i}+12$		
		$=0$		
		Thus, $-\mathrm{i}$ is a root of the equation.	AG	
				[2]
	(b)	The product of roots		
		$=\frac{12}{1}$		
		$=12$	A1	
				[1]
	(c)	The second root is i.	A1	
		Let $x^4-8x^3+13x^2-8x+12$		
		$=(x^2+bx+c)(x-\mathrm{i})(x-(-\mathrm{i}))$.		
		$x^4-8x^3+13x^2-8x+12=(x^2+bx+c)(x^2+1)$	M1	
		$x^4-8x^3+13x^2-8x+12$ $=x^4+bx^3+(c+1)x^2+bx+c$	A1	
		$\therefore b=-8$ and $c=12$		
		$x^2-8x+12=(x-2)(x-6)$		
		Thus, the other two roots are 2 and 6.	A2	
				[5]
2.	The third root is $a-b\mathrm{i}$.		A1	
	$4+(a+b\mathrm{i})+(a-b\mathrm{i})=-\frac{-12}{1}$		M1A1	
	$4+2a=12$			
	$2a=8$			
	$a=4$		A1	
	$(4)(4+b\mathrm{i})(4-b\mathrm{i})=-\frac{-68}{1}$		A1	
	$(4)(16+b^2)=68$			
	$16+b^2=17$			
	$b^2=1$			
	$b=-1$ (*Rejected*) or $b=1$		A1	
				[6]

3. The other two roots are $a-b\text{i}$ and $b-a\text{i}$. A1

$(a+b\text{i})+(a-b\text{i})+(b+a\text{i})+(b-a\text{i})=-\frac{-14}{1}$ M1A1

$2a+2b=14$

$b=7-a$

$(a+b\text{i})(a-b\text{i})(b+a\text{i})(b-a\text{i})=\frac{841}{1}$ A1

$(a^2+b^2)(b^2+a^2)=841$

$(a^2+b^2)^2=841$

$a^2+b^2=29$

$\therefore a^2+(7-a)^2=29$ M1

$a^2+49-14a+a^2=29$

$2a^2-14a+20=0$ A1

$2(a-2)(a-5)=0$

$a=2$ or $a=5$

$b=5$ (*Rejected*) or $b=2$

Thus, $a=5$ and $b=2$. A2

[8]

4. The second root is $a-b\text{i}$. A1

The third and the fourth root are 6 as the graph of $y=P(x)$ touches the x-axis at $(6, 0)$. A1

$(a+b\text{i})+(a-b\text{i})+6+6=-\frac{-16}{1}$ M1A1

$2a+12=16$

$2a=4$

$a=2$ A1

$(2+b\text{i})(2-b\text{i})(6)(6)=\frac{468}{1}$ A1

$(4+b^2)(36)=468$

$4+b^2=13$

$b^2=9$

$b=-3$ (*Rejected*) or $b=3$ A1

[7]

Exercise 39

1. (a) (i) $(\cos\theta + \mathrm{i}\sin\theta)^4$

$= \cos^4\theta + \binom{4}{1}\mathrm{i}\cos^3\theta\sin\theta$

$+\binom{4}{2}\mathrm{i}^2\cos^2\theta\sin^2\theta + \binom{4}{3}\mathrm{i}^3\cos\theta\sin^3\theta$ A1

$+\mathrm{i}^4\sin^4\theta$

$= \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta$

$-4\mathrm{i}\cos\theta\sin^3\theta + \sin^4\theta$ A1

(ii) $(\cos\theta + \mathrm{i}\sin\theta)^4 = \cos 4\theta + \mathrm{i}\sin 4\theta$ A1

$\cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta$

$-4\mathrm{i}\cos\theta\sin^3\theta + \sin^4\theta = \cos 4\theta + \mathrm{i}\sin 4\theta$ M1

$\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$+\mathrm{i}(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

$= \cos 4\theta + \mathrm{i}\sin 4\theta$

$\therefore \cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

and $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ A2

[6]

(b) $z^4 + 16\text{i} = 0$

$z^4 = -16\text{i}$

$z^4 = 2^4\left(\cos\frac{3\pi}{2} + \text{i}\sin\frac{3\pi}{2}\right)$ A1

$z = 2\left(\cos\left(\frac{\frac{3\pi}{2} + 2k\pi}{4}\right) + \text{i}\sin\left(\frac{\frac{3\pi}{2} + 2k\pi}{4}\right)\right)$ M1

$(k = 0, 1, 2, 3)$

$\therefore z = 2\left(\cos\left(\frac{\frac{3\pi}{2} + 2(0)\pi}{4}\right) + \text{i}\sin\left(\frac{\frac{3\pi}{2} + 2(0)\pi}{4}\right)\right)$

$z = 2\left(\cos\frac{3\pi}{8} + \text{i}\sin\frac{3\pi}{8}\right)$

$\therefore r = 2,\ \alpha = \frac{3\pi}{8}$ A2

[4]

(c) $\cos 4\left(\frac{3\pi}{8}\right) = \cos^4\frac{3\pi}{8} - 6\cos^2\frac{3\pi}{8}\sin^2\frac{3\pi}{8} + \sin^4\frac{3\pi}{8}$ M1

$0 = \cos^4\frac{3\pi}{8} - 6\cos^2\frac{3\pi}{8}\left(1-\cos^2\frac{3\pi}{8}\right)$

A1

$+\left(1-\cos^2\frac{3\pi}{8}\right)^2$

$0 = \cos^4\frac{3\pi}{8} - 6\cos^2\frac{3\pi}{8} + 6\cos^4\frac{3\pi}{8} + 1$

$-2\cos^2\frac{3\pi}{8} + \cos^4\frac{3\pi}{8}$

$0 = 8\cos^4\frac{3\pi}{8} - 8\cos^2\frac{3\pi}{8} + 1$ A1

$\cos^2\frac{3\pi}{8} = \frac{-(-8)\pm\sqrt{(-8)^2-4(8)(1)}}{2(8)}$ M1

$\cos^2\frac{3\pi}{8} = \frac{8\pm\sqrt{32}}{16}$

As $\frac{\pi}{3} < \frac{3\pi}{8} < \frac{\pi}{2}$, $0 < \cos\frac{3\pi}{8} < \frac{1}{2}$ and

$0 < \cos^2\frac{3\pi}{8} < \frac{1}{4}$.

$\therefore \cos^2\frac{3\pi}{8} = \frac{8+\sqrt{32}}{16}$ (*Rejected*) or

$\cos^2\frac{3\pi}{8} = \frac{8-\sqrt{32}}{16}$. A1

$\cos\frac{3\pi}{8} = \sqrt{\frac{8-4\sqrt{2}}{16}}$

$\cos\frac{3\pi}{8} = \sqrt{\frac{2-\sqrt{2}}{4}}$ A1

[6]

(d) $\sin\frac{3\pi}{8}\cos\frac{3\pi}{8}\left(\cos\frac{3\pi}{8}+\sin\frac{3\pi}{8}\right)\left(\cos\frac{3\pi}{8}-\sin\frac{3\pi}{8}\right)$

$=\sin\frac{3\pi}{8}\cos\frac{3\pi}{8}\left(\cos^2\frac{3\pi}{8}-\sin^2\frac{3\pi}{8}\right)$ M1

$=\cos^3\frac{3\pi}{8}\sin\frac{3\pi}{8}-\cos\frac{3\pi}{8}\sin^3\frac{3\pi}{8}$

$=\dfrac{4\cos^3\frac{3\pi}{8}\sin\frac{3\pi}{8}-4\cos\frac{3\pi}{8}\sin^3\frac{3\pi}{8}}{4}$ A1

$=\dfrac{\sin 4\left(\frac{3\pi}{8}\right)}{4}$ A1

$=-\frac{1}{4}$ AG

[3]

2. (a) $(\cos\theta + \mathrm{i}\sin\theta)^8$

$= \cos^8\theta + \binom{8}{1}\mathrm{i}\cos^7\theta\sin\theta + \binom{8}{2}\mathrm{i}^2\cos^6\theta\sin^2\theta$

$+\binom{8}{3}\mathrm{i}^3\cos^5\theta\sin^3\theta + \binom{8}{4}\mathrm{i}^4\cos^4\theta\sin^4\theta$

$+\binom{8}{5}\mathrm{i}^5\cos^3\theta\sin^5\theta + \binom{8}{6}\mathrm{i}^6\cos^2\theta\sin^6\theta$ A2

$+\binom{8}{7}\mathrm{i}^7\cos\theta\sin^7\theta + \mathrm{i}^8\sin^8\theta$

$= \cos^8\theta + 8\mathrm{i}\cos^7\theta\sin\theta - 28\cos^6\theta\sin^2\theta$

$-56\mathrm{i}\cos^5\theta\sin^3\theta + 70\cos^4\theta\sin^4\theta$

$+56\mathrm{i}\cos^3\theta\sin^5\theta - 28\cos^2\theta\sin^6\theta$ A1

$-8\mathrm{i}\cos\theta\sin^7\theta + \sin^8\theta$

$\therefore \cos 8\theta + \mathrm{i}\sin 8\theta = \cos^8\theta - 28\cos^6\theta\sin^2\theta$

$+70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta$

$+\mathrm{i}(8\cos^7\theta\sin\theta - 56\cos^5\theta\sin^3\theta$ M1

$+56\cos^3\theta\sin^5\theta - 8\cos\theta\sin^7\theta)$

$\sin 8\theta = 8\cos^7\theta\sin\theta - 56\cos^5\theta\sin^3\theta$

$+56\cos^3\theta\sin^5\theta - 8\cos\theta\sin^7\theta$ A1

[5]

(b) $z^8 + 6561 = 0$

$z^8 = -6561$

$z^8 = 3^8(\cos\pi + \mathrm{i}\sin\pi)$ A1

$z = 3\left(\cos\left(\dfrac{\pi + 2k\pi}{8}\right) + \mathrm{i}\sin\left(\dfrac{\pi + 2k\pi}{8}\right)\right)$ M1

$(k = 0, 1, 2, 3, 4, 5, 6, 7)$

$\therefore z_1 = 3\left(\cos\left(\dfrac{\pi + 2(7)\pi}{8}\right) + \mathrm{i}\sin\left(\dfrac{\pi + 2(7)\pi}{8}\right)\right)$

$z_1 = 3\left(\cos\dfrac{15\pi}{8} + \mathrm{i}\sin\dfrac{15\pi}{8}\right)$

$z_1 = 3\left(\cos\left(-\dfrac{\pi}{8}\right) + \mathrm{i}\sin\left(-\dfrac{\pi}{8}\right)\right)$

$\therefore r = 3,\ \alpha_1 = -\dfrac{\pi}{8}$ A2

[4]

$$\sin 8\left(-\frac{\pi}{8}\right)=8\cos^7\left(-\frac{\pi}{8}\right)\sin\left(-\frac{\pi}{8}\right)$$

(c) $-56\cos^5\left(-\frac{\pi}{8}\right)\sin^3\left(-\frac{\pi}{8}\right)$ M1

$+56\cos^3\left(-\frac{\pi}{8}\right)\sin^5\left(-\frac{\pi}{8}\right)-8\cos\left(-\frac{\pi}{8}\right)\sin^7\left(-\frac{\pi}{8}\right)$

$0=8\cos^7\alpha_1\sin\alpha_1-56\cos^5\alpha_1\sin^3\alpha_1$
$+56\cos^3\alpha_1\sin^5\alpha_1-8\cos\alpha_1\sin^7\alpha_1$ A1

$0=\cos^6\alpha_1-7\cos^4\alpha_1\sin^2\alpha_1$
$+7\cos^2\alpha_1\sin^4\alpha_1-\sin^6\alpha_1$ A1

$1-7\tan^2\alpha_1+7\tan^4\alpha_1-\tan^6\alpha_1=0$ AG

[3]

(d) (i) 3 A1

(ii) Rectangle A1

(iii) Trapezium A1

(iv) $z_2=3\left(\cos\left(\frac{\pi+2(6)\pi}{8}\right)+\mathrm{i}\sin\left(\frac{\pi+2(6)\pi}{8}\right)\right)$ M1

$z_2=3\left(\cos\frac{13\pi}{8}+\mathrm{i}\sin\frac{13\pi}{8}\right)$

$z_2=3\left(\cos\left(-\frac{3\pi}{8}\right)+\mathrm{i}\sin\left(-\frac{3\pi}{8}\right)\right)$

$\arg(z_2)=-\frac{3\pi}{8}$

$\therefore \arg(z_2^*)=\frac{3\pi}{8}$

$\mathrm{B\hat{O}E}=\frac{3\pi}{8}-\left(-\frac{3\pi}{8}\right)=\frac{3\pi}{4}$ A1

$\mathrm{BE}^2=3^2+3^2-2(3)(3)\cos\frac{3\pi}{4}$ A1

$\mathrm{BE}=\sqrt{9+9-18\cos\frac{3\pi}{4}}$

$\mathrm{BE}=5.543277195$

$\mathrm{BE}=5.54$ A1

[7]

3. (a) When $n=1$,

$\text{L.H.S.}=(-\cos\theta+\mathrm{i}\sin\theta)^2$

$\text{L.H.S.}=\cos^2\theta-2\mathrm{i}\sin\theta\cos\theta+\mathrm{i}^2\sin^2\theta$

$\text{L.H.S.}=\cos^2\theta-\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$

$\text{L.H.S.}=\cos 2\theta-\mathrm{i}\sin 2\theta$

$\text{R.H.S.}=\cos 2\theta-\mathrm{i}\sin 2\theta$

Thus, the statement is true when $n=1$. R1

Assume that the statement is true when $n=k$. M1

$(-\cos\theta+\mathrm{i}\sin\theta)^{2k}=\cos 2k\theta-\mathrm{i}\sin 2k\theta$

When $n=k+1$,

$(-\cos\theta+\mathrm{i}\sin\theta)^{2(k+1)}$

$=(-\cos\theta+\mathrm{i}\sin\theta)^{2k}(-\cos\theta+\mathrm{i}\sin\theta)^2$ M1

$(-\cos\theta+\mathrm{i}\sin\theta)^{2(k+1)}$

$=(\cos 2k\theta-\mathrm{i}\sin 2k\theta)(\cos 2\theta-\mathrm{i}\sin 2\theta)$ A1

$(-\cos\theta+\mathrm{i}\sin\theta)^{2(k+1)}=\cos 2k\theta\cos 2\theta$

$-\mathrm{i}\cos 2k\theta\sin 2\theta-\mathrm{i}\sin 2k\theta\cos 2\theta+\mathrm{i}^2\sin 2k\theta\sin 2\theta$

$(-\cos\theta+\mathrm{i}\sin\theta)^{2(k+1)}$

$=\cos 2k\theta\cos 2\theta-\sin 2k\theta\sin 2\theta$ A1

$-\mathrm{i}(\cos 2k\theta\sin 2\theta+\sin 2k\theta\cos 2\theta)$

$(-\cos\theta+\mathrm{i}\sin\theta)^{2(k+1)}$

$=\cos(2k\theta+2\theta)-\mathrm{i}\sin(2k\theta+2\theta)$

$(-\cos\theta+\mathrm{i}\sin\theta)^{2(k+1)}$

$=\cos 2(k+1)\theta-\mathrm{i}\sin 2(k+1)\theta$

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$. R1

[6]

(b) $(-\cos\theta+\mathrm{i}\sin\theta)^8$

$= (-\cos\theta)^8 + \binom{8}{1}\mathrm{i}(-\cos\theta)^7\sin\theta$

$+\binom{8}{2}\mathrm{i}^2(-\cos\theta)^6\sin^2\theta + \binom{8}{3}\mathrm{i}^3(-\cos\theta)^5\sin^3\theta$

$+\binom{8}{4}\mathrm{i}^4(-\cos\theta)^4\sin^4\theta + \binom{8}{5}\mathrm{i}^5(-\cos\theta)^3\sin^5\theta$ A2

$+\binom{8}{6}\mathrm{i}^6(-\cos\theta)^2\sin^6\theta + \binom{8}{7}\mathrm{i}^7(-\cos\theta)\sin^7\theta$

$+\mathrm{i}^8\sin^8\theta$

$= \cos^8\theta - 8\mathrm{i}\cos^7\theta\sin\theta - 28\cos^6\theta\sin^2\theta$

$+56\mathrm{i}\cos^5\theta\sin^3\theta + 70\cos^4\theta\sin^4\theta$ A1

$-56\mathrm{i}\cos^3\theta\sin^5\theta - 28\cos^2\theta\sin^6\theta$

$+8\mathrm{i}\cos\theta\sin^7\theta + \sin^8\theta$

$\therefore \cos 8\theta - \mathrm{i}\sin 8\theta = \cos^8\theta - 28\cos^6\theta\sin^2\theta$

$+70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta$ M1

$+\mathrm{i}(-8\cos^7\theta\sin\theta + 56\cos^5\theta\sin^3\theta$

$-56\cos^3\theta\sin^5\theta + 8\cos\theta\sin^7\theta)$

$\cos 8\theta = \cos^8\theta - 28\cos^6\theta\sin^2\theta$ A1

$+70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta$

[5]

(c) $\cos 8\theta = 41\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta$

$\therefore \cos^8\theta - 28\cos^6\theta\sin^2\theta$

$+70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta$ M1

$= 41\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta$

$\cos^8\theta - 28\cos^6\theta\sin^2\theta + 29\cos^4\theta\sin^4\theta = 0$

$\cos^4\theta - 28\cos^2\theta\sin^2\theta + 29\sin^4\theta = 0$ A1

$(\cos^2\theta)^2 - 28\cos^2\theta\sin^2\theta + 29(\sin^2\theta)^2 = 0$

$(\cos^2\theta - \sin^2\theta)(\cos^2\theta - 28\sin^2\theta) = 0$ A1

$\cos^2\theta - \sin^2\theta = 0$ or $\cos^2\theta - 28\sin^2\theta = 0$

$\tan^2\theta = 1$ or $\tan^2\theta = \frac{1}{28}$ A1

$\tan\theta = -1$ (*Rejected*), $\tan\theta = -\frac{1}{\sqrt{28}}$ (*Rejected*),

$\tan\theta = \frac{1}{\sqrt{28}}$ or $\tan\theta = 1$

$\theta = 0.186779461$ or $\theta = \frac{\pi}{4}$

$\theta = 0.187$ or $\theta = \frac{\pi}{4}$ A2

[6]

4. (a) $\left(\cos\frac{\theta}{5}+i\sin\frac{\theta}{5}\right)^5$

$=\cos^5\frac{\theta}{5}+\binom{5}{1}i\cos^4\frac{\theta}{5}\sin\frac{\theta}{5}+\binom{5}{2}i^2\cos^3\frac{\theta}{5}\sin^2\frac{\theta}{5}$

$+\binom{5}{3}i^3\cos^2\frac{\theta}{5}\sin^3\frac{\theta}{5}+\binom{5}{4}i^4\cos\frac{\theta}{5}\sin^4\frac{\theta}{5}$ A2

$+i^5\sin^5\frac{\theta}{5}$

$=\cos^5\frac{\theta}{5}+5i\cos^4\frac{\theta}{5}\sin\frac{\theta}{5}-10\cos^3\frac{\theta}{5}\sin^2\frac{\theta}{5}$

A1

$-10i\cos^2\frac{\theta}{5}\sin^3\frac{\theta}{5}+5\cos\frac{\theta}{5}\sin^4\frac{\theta}{5}+i\sin^5\frac{\theta}{5}$

$\therefore \cos\theta+i\sin\theta$

$=\cos^5\frac{\theta}{5}+5i\cos^4\frac{\theta}{5}\sin\frac{\theta}{5}-10\cos^3\frac{\theta}{5}\sin^2\frac{\theta}{5}$ M1

$-10i\cos^2\frac{\theta}{5}\sin^3\frac{\theta}{5}+5\cos\frac{\theta}{5}\sin^4\frac{\theta}{5}+i\sin^5\frac{\theta}{5}$

$\cos\theta+i\sin\theta$

$=\cos^5\frac{\theta}{5}-10\cos^3\frac{\theta}{5}\sin^2\frac{\theta}{5}+5\cos\frac{\theta}{5}\sin^4\frac{\theta}{5}$

$+i\left(5\cos^4\frac{\theta}{5}\sin\frac{\theta}{5}-10\cos^2\frac{\theta}{5}\sin^3\frac{\theta}{5}+\sin^5\frac{\theta}{5}\right)$

$\therefore \cos\theta=\cos^5\frac{\theta}{5}-10\cos^3\frac{\theta}{5}\sin^2\frac{\theta}{5}+5\cos\frac{\theta}{5}\sin^4\frac{\theta}{5}$

and $\sin\theta=5\cos^4\frac{\theta}{5}\sin\frac{\theta}{5}-10\cos^2\frac{\theta}{5}\sin^3\frac{\theta}{5}+\sin^5\frac{\theta}{5}$. A2

[6]

(b) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$\tan\theta = \frac{5\cos^4\frac{\theta}{5}\sin\frac{\theta}{5} - 10\cos^2\frac{\theta}{5}\sin^3\frac{\theta}{5} + \sin^5\frac{\theta}{5}}{\cos^5\frac{\theta}{5} - 10\cos^3\frac{\theta}{5}\sin^2\frac{\theta}{5} + 5\cos\frac{\theta}{5}\sin^4\frac{\theta}{5}}$$ M1A1

$$\tan\theta = \frac{5\tan\frac{\theta}{5} - 10\tan^3\frac{\theta}{5} + \tan^5\frac{\theta}{5}}{1 - 10\tan^2\frac{\theta}{5} + 5\tan^4\frac{\theta}{5}}$$ A1

$$\tan\theta = \frac{\tan\frac{\theta}{5}\left(5 - 10\tan^2\frac{\theta}{5} + \tan^4\frac{\theta}{5}\right)}{1 - 10\tan^2\frac{\theta}{5} + 5\tan^4\frac{\theta}{5}}$$ AG

[3]

(c) $x = \tan\dfrac{\theta}{5}$

$$\tan\theta = \frac{x(5 - 10x^2 + x^4)}{1 - 10x^2 + 5x^4}$$ M1

$x^4 - 10x^2 + 5 = 0$

$$\frac{x(5 - 10x^2 + x^4)}{1 - 10x^2 + 5x^4} = 0$$

$\tan\theta = 0$ M1

$\theta = 0,\ \pi,\ 2\pi,\ 3\pi$ or 4π

$\therefore x = \tan\dfrac{0}{5},\ x = \tan\dfrac{\pi}{5},\ x = \tan\dfrac{2\pi}{5},\ x = \tan\dfrac{3\pi}{5}$ or

$x = \tan\dfrac{4\pi}{5}$ A1

$x = 0$ (*Rejected*), $x = \tan\dfrac{\pi}{5}$, $x = \tan\dfrac{2\pi}{5}$,

$x = \tan\dfrac{3\pi}{5}$ or $x = \tan\dfrac{4\pi}{5}$ A1

[4]

(d) (i) $\tan\frac{\pi}{5}+\tan\frac{2\pi}{5}+\tan\frac{3\pi}{5}+\tan\frac{4\pi}{5}=-\frac{0}{1}$ M1A1

$\tan\frac{0\pi}{5}+\tan\frac{\pi}{5}+\tan\frac{2\pi}{5}+\tan\frac{3\pi}{5}$

$+\tan\frac{4\pi}{5}=0$

$\therefore \sum_{r=0}^{4}\tan\frac{r\pi}{5}=0$ A1

(ii) $\left(\tan\frac{\pi}{5}\right)\left(\tan\frac{2\pi}{5}\right)\left(\tan\frac{3\pi}{5}\right)\left(\tan\frac{4\pi}{5}\right)=\frac{5}{1}$ M1A1

$\left(\tan\frac{\pi}{5}\right)\left(\tan\frac{2\pi}{5}\right)\left(\tan\left(\pi-\frac{2\pi}{5}\right)\right)$

$\left(\tan\left(\pi-\frac{\pi}{5}\right)\right)=5$

$\left(\tan\frac{\pi}{5}\right)\left(\tan\frac{2\pi}{5}\right)\left(-\tan\frac{2\pi}{5}\right)\left(-\tan\frac{\pi}{5}\right)=5$ A1

$\left(\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\right)^2=5$

$\tan\frac{\pi}{5}\tan\frac{2\pi}{5}=\sqrt{5}$ A1

[7]