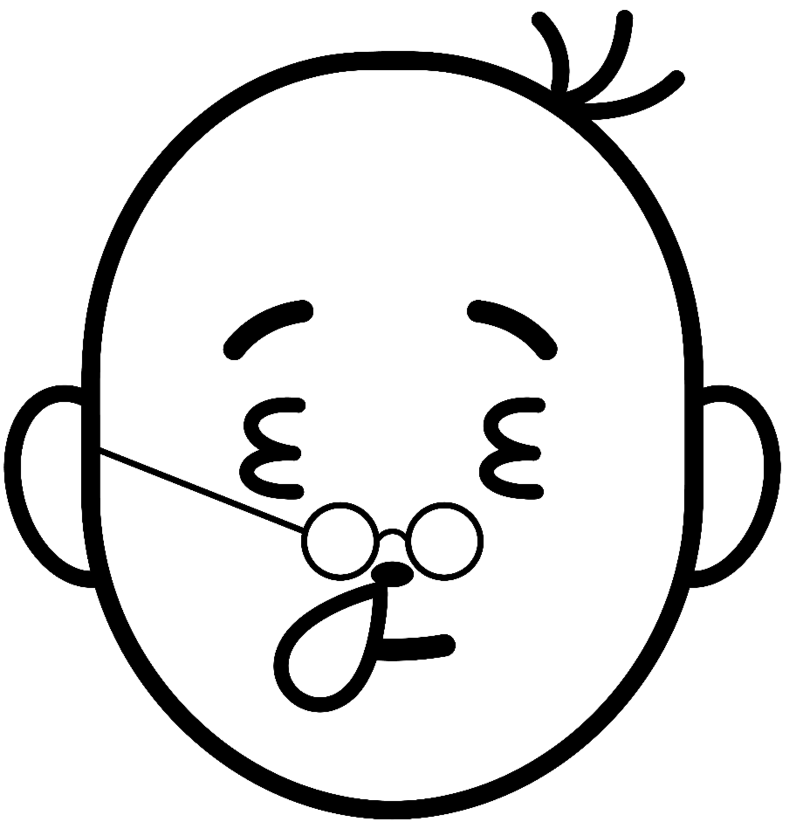
Your Intensive Notes Applications and Interpretation Higher Level for IBDP Mathematics



Algebra

Topics Covered

1	Standard Form	Page 3
2	Approximation and Error	Page 5
3	Arithmetic Sequences	Page 8
4	Geometric Sequences	Page 13
5	Systems of Equations	Page 19
6	Financial Mathematics	Page 21
7	Complex Numbers	Page 31
8	Matrices	Page 39

1 Standard Form

Important Notes

Standard form: A number in the format $\pm a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$ (k is an integer)

Notes on GDC		
TEXAS TI-84 Plus CE	**TEXAS TI-Nspire CX**	**CASIO fx-CG50**
mode→SCI on the 2nd row to express any number in its standard form	Doc→Setting & Status →Document Settings… →Scientific on the Exponential Format row to express any number in its standard form	SHIFT MENU →Sci on the Display row →3 to express any number in its standard form

Example 1.1

A rectangle is 2376 cm long and 693 cm wide.

(a) Find the diagonal length of the rectangle, giving the answer in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$.

[2]

The required diagonal length

$= \sqrt{2376^2 + 693^2}$ Pythagoras' theorem (A1)

$= 2475 \text{ cm}$

$= 2.475 \times 10^3 \text{ cm}$ $a = 2.475$ & $k = 3$ (A1)

(b) Find the area of the rectangle, giving the answer correct to two significant figures and in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$.

[2]

The required area

$= (2376)(693)$ Base Length $\times$ Height (A1)

$= 1646568 \text{ cm}^2$

$= 1600000 \text{ cm}^2$ Round off to 2 sig. fig.

$= 1.6 \times 10^6 \text{ cm}^2$ $a = 1.6$ & $k = 6$ (A1)

Exercise 1.1

The base length and the height of a right-angled triangle are 1107 mm and 4920 mm respectively.

(a) Find the length of the hypotenuse of the triangle, giving the answer in the form $a\times 10^{k}$, where $1\le a<10$ and $k\in\mathbb{Z}$.

[2]

(b) Using (a), find the perimeter of the triangle, giving the answer correct to two significant figures and in the form $a\times 10^{k}$, where $1\le a<10$ and $k\in\mathbb{Z}$.

[2]

(c) Find the area of the triangle, giving the answer correct to four significant figures and in the form $a\times 10^{k}$, where $1\le a<10$ and $k\in\mathbb{Z}$.

[2]

2 Approximation and Error

Important Notes

Common rounding methods:

1. $2.7\overset{+1}{1}\underline{8}28 \to 2.72$ (Correct to 3 significant figures)
2. $2.718\underline{2}8 \to 2.718$ (Correct to 3 decimal places)
3. $\overset{+1}{2}.\underline{7}1828 \to 3$ (Correct to the nearest integer)

Exact and approximated values:

1. v_E : Exact value
2. v_A : Approximated value corrected to the nearest unit d
3. $\frac{1}{2}d$: Maximum absolute error
4. $v_A - \frac{1}{2}d$: Lower bound (Least possible value) of v_E
5. $v_A + \frac{1}{2}d$: Upper bound of v_E
6. $v_A - \frac{1}{2}d \le v_E < v_A + \frac{1}{2}d$: Range of v_E
7. $\left|\frac{v_A - v_E}{v_E}\right| \times 100\%$: Percentage error

Example 1.2

In a charity booth, there is a transparent box completely filled with identical cubic blocks. Participants have to estimate the number of cubic blocks in the box. The box is 30 cm long, 30 cm wide and 20 cm tall.

(a) Find the volume of the box.

[2]

The volume

$=(30)(30)(20)$ — $V=lwh$ (M1)

$=18000\text{ cm}^3$ — (A1)

Sergio estimates the volume of one cubic block to be 200 cm^3 and uses this value to estimate the number of cubic blocks in the box.

(b) Find Sergio's estimated number of cubes in the box.

[2]

The estimated number

$=\dfrac{18000}{200}$ — Divided by 200 (M1)

$=90$ — (A1)

The actual number of cubic blocks in the box is 87.

(c) Find the percentage error in Sergio's estimated number of cubic blocks in the box.

[2]

The percentage error

$=\left|\dfrac{90-87}{87}\right|\times 100\%$ — $\left|\dfrac{v_A-v_E}{v_E}\right|\times 100\%$ (M1)

$=3.448275862\%$

$=3.45\%$ — (A1)

Exercise 1.2

A delivery container is a cuboid with dimensions 2 m, 0.8 m and 0.8 m.

(a) Find the exact volume of the container.

[2]

Zac estimates the dimensions of the container as 2.2 m, 1 m and 0.7 m, and uses these to estimate the volume of the container.

(b) Find the percentage error in Zac's estimated volume of the container.

[3]

3 Arithmetic Sequences

Important Notes

An arithmetic sequence is a sequence such that the next term is generated by adding or subtracting the same number from the previous term.

Properties of an arithmetic sequence $u_1, u_2, u_3, \cdots$:

1. u_1: First term
2. $d = u_2 - u_1 = u_n - u_{n-1}$: Common difference
3. $u_n = u_1 + (n-1)d$: General term (nth term)
4. $S_n = \frac{n}{2}[2u_1 + (n-1)d] = \frac{n}{2}[u_1 + u_n]$: Sum of the first n terms

$\sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + \cdots + u_{n-1} + u_n$: Summation sign

Notes on GDC		
TEXAS TI-84 Plus CE [y=] to input the general term →[2nd] [window] to set the starting row →[2nd] [graph] to find the value of the term needed	**TEXAS TI-Nspire CX** Graph to input the general term to generate a table →[ctrl] [T] to generate a table →[menu] [2] [5] to set the starting row to find the value of the term needed	**CASIO fx-CG50** Table to input the general term to generate a table →[F5] to set the starting row →[F6] to find the value of the term needed

Example 1.3

Mady designs a decorative glass face for a museum. The glass face is made up of small triangular panes. The first level, at the bottom of the glass face, has seven triangular panes. The second level has nine triangular panes, and the third level has eleven triangular panes. Each additional level has two more triangular panes than the level below it.

(a) Find the number of triangular panes in the 10th level.

[3]

The number of triangular panes

$= u_{10}$ — 10th term (M1)

$= 7 + (10-1)(2)$ — $u_1 = 7$ & $d = 2$ (A1)

$= 25$ — (A1)

It is given that there are 41 triangular panes in the rth level.

(b) Find r.

[2]

$u_r = 41$ — Set up an equation

$\therefore 7 + (r-1)(2) = 41$ — Correct equation (A1)

$2(r-1) = 34$

$r - 1 = 17$

$r = 18$ — (A1)

(c) (i) Show that the total number of triangular panes S_n in the first n levels is given by $S_n = n^2 + 6n$.

[3]

S_n

$= \frac{n}{2}\left[2u_1 + (n-1)d\right]$ — $S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$ (M1)

$= \frac{n}{2}\left[2(7) + (n-1)(2)\right]$ — $u_1 = 7$ & $d = 2$ (A1)

$= \frac{n}{2}(14 + 2n - 2)$

$= \frac{n(2n+12)}{2}$

$= \frac{2n^2 + 12n}{2}$ — $2n^2 + 12n$ (M1)

$= n^2 + 6n$ — (AG)

(ii) Hence, find the total number of panes in a glass face with 13 levels. [2]

The total number of panes

$= S_{13}$

$= 13^2 + 6(13)$ $n = 13$ (M1)

$= 247$ (A1)

Mady has 500 triangular panes to build the decorative glass face and does not want it to have any incomplete levels.

(d) Find the greatest number of complete levels that Mady can build. [3]

$S_n = 500$

$\therefore n^2 + 6n = 500$ Correct equation (A1)

$n^2 + 6n - 500 = 0$

By considering the graph of $y = n^2 + 6n - 500$,

the horizontal intercepts are -25.56103

(*Rejected*) and 19.561028. GDC approach (M1)

Thus, the greatest number of complete levels that Mady can build is 19. (A1)

Each triangular pane has an area of $2.27\ \text{m}^2$ and the maximum number of complete levels were built.

(e) Find the total area of the decorative glass face, giving the area to the nearest m^2. [3]

The total area

$= S_{19} \times 2.27$ Multiply by 2.27 (M1)

$= \left[19^2 + 6(19)\right](2.27)$ $n = 19$ (M1)

$= 1078.25\ \text{m}^2$

$= 1078\ \text{m}^2$ (A1)

Exercise 1.3

In a party game, a number of apples are placed one metre apart in a straight line. Players are gathered at the starting position which is five metres before the first apple.

Each player collects a single apple by picking it up and bringing it back to the starting position. The nearest apple is collected first. The player then collects the first nearest apple and the game continues in this way until the signal is given for the end.

Fatima runs to get each apple and brings it back to the start. Let u_1 metres be the distance she has to run in order to collect the first apple – that is, to pick up the first apple and bring it back to the starting position.

(a) Write down u_1.

[1]

(b) (i) Show that $u_2 = 12$ and $u_3 = 14$.

[1]

(ii) Hence, write down the common difference d of the sequence.

[1]

The final apple Fatima collected was 20 metres from the starting position.

(c) (i) Find the total number of apples that Fatima collected.

[2]

(ii) Find the total distance that Fatima ran to collect these apples.

[3]

Akash also plays the game. When the signal is given for the end of the game he has run 491 metres.

(d) (i) Find the total number of apples that Akash has collected.

[3]

(ii) Hence, find the distance between Akash and the starting position when the signal is given.

[3]

4 Geometric Sequences

Important Notes

A geometric sequence is a sequence such that the next term is generated by multiplying or being divided by the same number from the previous term.

Properties of a geometric sequence $u_1, u_2, u_3, \cdots$:

1. u_1: First term
2. $r = u_2 \div u_1 = u_n \div u_{n-1}$: Common ratio
3. $u_n = u_1 \times r^{n-1}$: General term (nth term)
4. $S_n = \frac{u_1(1-r^n)}{1-r} = \frac{u_1(r^n-1)}{r-1}$: Sum of the first n terms
5. $S_\infty = u_1 + u_2 + u_3 + \cdots + u_n + \cdots = \frac{u_1}{1-r}$: Sum of infinite number of terms (Sum to infinity), valid only when $-1 < r < 1$

Notes on GDC		
TEXAS TI-84 Plus CE	**TEXAS TI-Nspire CX**	**CASIO fx-CG50**
[y=] to input the general term →[2nd] [window] to set the starting row →[2nd] [graph] to find the value of the term needed	Graph to input the general term to generate a table →[ctrl] [T] to generate a table →[menu] [2] [5] to set the starting row to find the value of the term needed	Table to input the general term to generate a table →[F5] to set the starting row →[F6] to find the value of the term needed

Example 1.4

On Monday, Peter goes to a running track to train. He runs the first lap of the track in 100 seconds. Each lap Peter runs takes 1.07 times as long as his previous lap.

(a) Find the time, in seconds, Peter takes to run his fourth lap. [3]

The required time

$= u_4$ 4th term (M1)

$= u_1 \times r^{4-1}$

$= 100 \times 1.07^3$ $u_1 = 100$ & $r = 1.07$ (A1)

$= 122.5043 \text{ s}$

$= 123 \text{ s}$ (A1)

Peter runs his last lap in at least 183 seconds.

(b) (i) Find the least possible number of laps he has run on Monday. [3]

$u_n \geq 183$ Set up an inequality

$\therefore 100 \times 1.07^{n-1} \geq 183$ Correct inequality (A1)

$100 \times 1.07^{n-1} - 183 \geq 0$

By considering the graph of $y = 100 \times 1.07^{n-1} - 183$, the graph is above the horizontal axis when $n > 9.9318362$. GDC approach (M1)

$\therefore$ The least possible number of laps he has run on Monday is 10. (A1)

(ii) Hence, find the total time, in minutes, run by Peter on Monday.

[4]

The total time

$= S_{10}$

$= \dfrac{u_1(1-r^{10})}{1-r}$ — $S_n = \dfrac{u_1(1-r^n)}{1-r}$ (M1)

$= \dfrac{(100)(1-1.07^{10})}{1-1.07}$ — $u_1 = 100$ & $r = 1.07$ (A1)

$= 1381.644796 \text{ s}$

$= \dfrac{1381.644796}{60} \text{ min}$ — $1 \text{ s} = \dfrac{1}{60} \text{ min}$ (A1)

$= 23.02741327 \text{ min}$

$= 23.0 \text{ min}$ — (A1)

On Tuesday, Peter takes Zoey to train. They both run the first lap of the track in 100 seconds. Each lap Zoey runs takes her 15 seconds longer than her previous lap.

(c) Find the time, in seconds, Zoey takes to run her fifth lap.

[3]

The required time

$= v_5$ — 5th term (M1)

$= 100 + (5-1)(15)$ — $v_1 = 100$ & $d = 15$ (A1)

$= 160 \text{ s}$ — (A1)

After a certain number of laps, Peter takes more time per lap than Zoey.

(d) Find the number of the lap when this happens.

[3]

$u_n > v_n$ — Set up an inequality

$\therefore 100 \times 1.07^{n-1} > 100 + (n-1)(15)$ — Correct inequality (A1)

$100 \times 1.07^{n-1} - 100 - 15(n-1) > 0$

By considering the graph of $y = 100 \times 1.07^{n-1} - 100 - 15(n-1)$, the graph is above the horizontal axis when $n < 1$ (*Rejected*) or $n > 22.072137$. — GDC approach (M1)

$\therefore$ Peter takes more time per lap than Zoey during the 23rd lap. — (A1)

On Wednesday, Jocelyn join Peter and Zoey to train. Three of them run the first lap of the track in 100 seconds. Each lap Jocelyn runs takes 80% times as long as her previous lap. Jocelyn claims that the total time spent on all her laps on that day could be at least 550 seconds.

(e) Explain why Jocelyn's claim is incorrect. [3]

The upper limit of the total time

$= S_\infty$

$= \dfrac{u_1}{1-r}$ $\quad S_\infty = \dfrac{u_1}{1-r}$ (M1)

$= \dfrac{100}{1-0.8}$ $\quad u_1 = 100$ & $r = 0.8$ (A1)

$= 500\text{ s}$

$< 550\text{ s}$ $\quad 500\text{ s} < 550\text{ s}$ (R1)

Thus, Jocelyn's claim is incorrect. (AG)

Exercise 1.4

A new café opened and during the first week their profit was $\$1200$. The café's profit increases by 8% every week.

(a) Find the café's profit during the tenth week.

[3]

(b) (i) Show that the café's **total** profit for the first n weeks is given by $S_n = -15000(1-1.08^n)$.

[2]

(ii) Hence, find the café's **total** profit for the first eleven weeks.

[2]

A new dessert shop opened at the same time as the café. During the first week their profit was also $\$1200$. The dessert shop's profit increases by $\$180$ every week.

(c) Find the dessert shop's profit during the seventh week.

[3]

In the m th week, the café's **total** profit exceeds the dessert shop's **total** profit for the first time since they both opened.

(d) Find m.

[3]

A new tea shop also opened at the same time as the café. During the first week their profit was also $\$1200$. The tea shop's profit decreases by 5% every week. The owner of the tea shop claims that it is possible for the tea shop's **total** profit to reach $\$22000$.

(e) Explain why the owner's claim is correct.

[3]

5 Systems of Equations

Important Notes

Common systems of equations:

1. $\begin{cases} ax+by=c \\ dx+ey=f \end{cases}$: 2×2 system, $a, b, c, d, e, f \in \mathbb{R}$

2. $\begin{cases} ax+by+cz=d \\ ex+fy+gz=h \\ ix+jy+kz=l \end{cases}$: 3×3 system, $a, b, c, d, e, f, g, h, i, j, k, l \in \mathbb{R}$

Example 1.5

The total revenue $\$P$ for selling x boxes of chocolate milk, y boxes of coffee milk and z boxes of oat milk can be modelled by $P=6x+5y+9z$, where $\$6$, $\$5$ and $\$9$ are the prices of a box of chocolate milk, coffee milk and oat milk respectively. It is given that

- The total number of boxes of milk sold is 28.
- The total revenue is $\$238$.
- There are twice as many boxes of coffee milk as boxes of chocolate milk.

(a) (i) Write down the system of three equations in x, y and z.

[3]

$\begin{cases} x+y+z=28 \\ 6x+5y+9z=238 \\ 2x-y=0 \end{cases}$ (A1)(A1)(A1)

(ii) Hence, find the values of x, y and z.

[3]

$x=7$, $y=14$ and $z=7$ (A1)(A1)(A1)

(b) Write down the total price of three boxes of oat milk and ten boxes of chocolate milk.

[1]

$\$87$ (A1)

Exercise 1.5

In a softball league, each team gains x points for a win, y points for a draw and z points for a loss. There are twenty matches in a year. The following table shows the performances of three teams in 2023:

Team	Win	Draw	Loss	Points
A	8	7	5	41
B	6	4	10	22
C	13	7	0	66

(a) (i) Write down the system of three equations in x, y and z.

[3]

(ii) Hence, find the values of x, y and z.

[3]

(b) Interpret the meaning of z.

[1]

6 Financial Mathematics

Important Notes

Properties of compound interest:

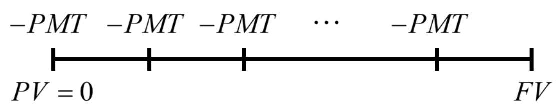
1. PV : Present value
2. $r\%$: Nominal annual interest rate
3. k : Number of compounded periods in one year
4. n : Number of years
5. $FV = PV\left(1+\frac{r\%}{k}\right)^{kn}$: Future value
6. $I = FV - PV$: Interest

Properties of inflation rates:

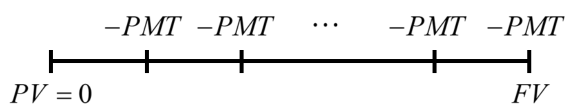
1. $i\%$: Inflation rate per year
2. $r\%$: Nominal annual interest rate
3. $(r-i)\%$: Real rate

Types of annuities:

1. Payments at the beginning of each time period

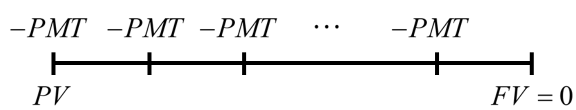


2. Payments at the end of each time period

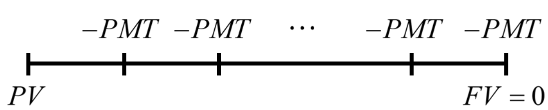


Types of amortizations:

1. Payments at the beginning of each time period



2. Payments at the end of each time period



Notes on GDC		
TEXAS TI-84 Plus CE N: Number of time periods I%: Annual interest rate PV: Present value PMT: Payment amount FV: Future value P/Y: Number of annual payments C/Y: Number of compounded periods (equals to P/Y) PMT END/BEGIN: Time of payment in each time period →[alpha] [enter] to solve the unknown quantity	**TEXAS TI-Nspire CX** N: Number of time periods I%: Annual interest rate PV: Present value Pmt: Payment amount FV: Future value PpY: Number of annual payments CpY: Number of compounded periods (equals to P/Y) PmtAt END/BEGIN: Time of payment in each time period →[enter] to solve the unknown quantity	**CASIO fx-CG50** n: Number of time periods I%: Annual interest rate PV: Present value PMT: Payment amount FV: Future value P/Y: Number of annual payments C/Y: Number of compounded periods (equals to P/Y) →[SHIFT] [MENU] to choose payment at the beginning/end of each time period →[F1] to [F5] to solve the unknown quantity

Example 1.6

On 1st January 2024, Judy invests $\$22000$ in an account that pays a nominal annual interest rate of 6%, compounded **half-yearly**. It is given that there is no further deposit to or any withdrawal from the account.

(a) (i) Find the amount that Judy will have in her account after 3 years. [3]

By financial solver:

$\text{N(n)} = 6$
$\text{I\%} = 6$
$\text{PV} = -22000$
$\text{PMT(Pmt)} = 0$
$\text{FV} = ?$
$\text{P/Y(PpY)} = 2$
$\text{C/Y(CpY)} = 2$
$\text{PMT(PmtAt)}: \text{END}$

GDC approach (M1)(A1)

$\text{FV} = 26269.15052$

Thus, the amount after 3 years is $\$26300$. (A1)

(ii) Hence, find the interest that Judy can earn after 3 years. [2]

The interest

$= 26269.15052 - 22000$ $\quad I = FV - PV$ (M1)

$= \$4269.150524$

$= \$4270$ (A1)

(b) Find the year in which the amount of money in Judy's account will become double the amount she invested.

[4]

By financial solver:

$N(n)=?$
$I\%=6$
$PV=-22000$
$PMT(Pmt)=0$
$FV=44000$
$P/Y(PpY)=2$
$C/Y(CpY)=2$
$PMT(PmtAt):END$

GDC approach (M1)(A1)

$N=23.44977225$

The number of years

$=\dfrac{23.44977225}{2}$

$=11.72488613$ $\quad$ 11.72488613 (A1)

Thus, the required year is 2035. (A1)

It is given that the rate of inflation during these 3 years is 2% per year.

(c) (i) Write down the value of the real interest rate.

[1]

4% (A1)

(ii) Hence, find the real value of the amount that Judy will have in her account after 3 years.

[3]

By financial solver:

$N(n)=6$
$I\%=4$
$PV=-22000$
$PMT(Pmt)=0$
$FV=?$
$P/Y(PpY)=2$
$C/Y(CpY)=2$
$PMT(PmtAt):END$

GDC approach (M1)(A1)

$FV=24775.57322$

Thus, the real amount after 3 years is $\$24800$. (A1)

Exercise 1.6

On 1st January 2025, April invests $\$10000$ in an account that pays a nominal annual interest rate of 8%, compounded **quarterly**. It is given that there is no further deposit to or any withdrawal from the account.

(a) (i) Find the amount that April will have in her account after 4 years. [3]

(ii) Hence, find the interest that April can earn after 4 years. [2]

(b) Find the year in which the amount of money in April's account will become 2.5 times the amount she invested. [4]

It is given that the rate of inflation during these 4 years is 3% per year.

(c) (i) Write down the value of the real interest rate.

[1]

(ii) Hence, find the real value of the amount that April will have in her account after 4 years.

[3]

Example 1.7

Takumi is going to purchase a boat. He is suggested to choose one of the two options to repay the loan of $\$2100000$:

Option 1: A total of 144 equal monthly payments have to be paid at the end of each month, with a nominal annual interest rate of 6%, compounded monthly
Option 2: A deposit of $\$400000$ has to be paid at the beginning of the loan, followed by monthly payments of $\$20000$ at the end of each month until the loan is fully repaid, with a nominal annual interest rate of 5.4%, compounded monthly

(a) If Takumi selects the option 1, find

(i) the amount of monthly payment, [3]

By financial solver:

$\text{N(n)} = 144$
$\text{I\%} = 6$
$\text{PV} = 2100000$
$\text{PMT(Pmt)} = ?$
$\text{FV} = 0$
$\text{P / Y(PpY)} = 12$
$\text{C / Y(CpY)} = 12$
$\text{PMT(PmtAt)} : \text{END}$

GDC approach (M1)(A1)

$\text{PMT} = -20492.85449$

Thus, the amount of monthly payment is
$\$20500$. (A1)

(ii) the total amount to be paid, [2]

The total amount
$= (20492.85449)(144)$ $PMT \times N$ (A1)
$= \$2950971.047$
$= \$2950000$ (A1)

(iii) the amount of interest paid. [2]

The amount of interest
$= 2950971.047 - 2100000$ $I = FV - PV$ (M1)
$= \$850971.0466$
$= \$851000$ (A1)

(b) If Takumi selects the option 2, find

(i) the number of months to repay the loan,

[3]

By financial solver:

$\text{N(n)} = ?$
$\text{I\%} = 5.4$
$\text{PV} = 1700000$
$\text{PMT(Pmt)} = -20000$
$\text{FV} = 0$
$\text{P / Y(PpY)} = 12$
$\text{C / Y(CpY)} = 12$
$\text{PMT(PmtAt)} : \text{END}$

GDC approach (M1)(A1)

$\text{N} = 107.3689045$

Thus, the required number of months is 108. (A1)

(ii) the exact total amount to be paid,

[2]

The exact total amount

$= 400000 + (20000)(108)$ $400000 + PMT \times N$ (A1)

$= \$2560000$ (A1)

(iii) the amount of interest paid.

[2]

The amount of interest

$= 2560000 - 2100000$ $I = FV - PV$ (M1)

$= \$460000$ (A1)

(c) By considering the amounts of interest paid in both options in both options, state the better option for Takumi and explain the answer.

[2]

The amount of interest paid in option 2 is less than that in option 1. Comparing interest (R1)

Thus, the option 2 is better. (A1)

Exercise 1.7

Jun Ryeol is going to buy a flat. He is suggested to choose one of the two options to repay the loan of 300000 USD with a nominal annual interest rate of 3.3%, compounded monthly:

Option 1: A total of 270 equal monthly payments have to be paid at the end of each month
Option 2: A monthly payment of 2250 USD has to be paid at the end of each month until the loan is fully repaid

(a) If Jun Ryeol selects the option 1, find

(i) the amount of monthly payment, [3]

(ii) the total amount to be paid, [2]

(iii) the amount of interest paid. [2]

(b) If Jun Ryeol selects the option 2, find

(i) the number of months to repay the loan, [3]

(ii) the exact total amount to be paid, [2]

(iii) the amount of interest paid. [2]

(c) By considering the amounts of monthly payment in both options, state the better option for Jun Ryeol and explain the answer. [2]

7 Complex Numbers

Important Notes

Terminologies of complex numbers:

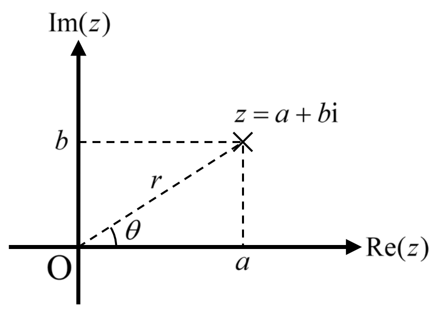
1. $\mathrm{i}=\sqrt{-1}$: Imaginary unit
2. $z=a+b\mathrm{i}$: Complex number in Cartesian form
3. $a=\mathrm{Re}(z)$: Real part of $z=a+b\mathrm{i}$
4. $b=\mathrm{Im}(z)$: Imaginary part of $z=a+b\mathrm{i}$
5. $z^*=a-b\mathrm{i}$: Conjugate of $z=a+b\mathrm{i}$
6. $|z|=\sqrt{a^2+b^2}$: Modulus of $z=a+b\mathrm{i}$
7. $\arg(z)=\arctan\dfrac{b}{a}$: Argument of $z=a+b\mathrm{i}$

Properties of i for $N\in\mathbb{Z}$:

1. $\mathrm{i}=\mathrm{i}^5=\mathrm{i}^9=\cdots=\mathrm{i}^{4N+1}=\mathrm{i}$
2. $\mathrm{i}^2=\mathrm{i}^6=\mathrm{i}^{10}=\cdots=\mathrm{i}^{4N+2}=-1$
3. $\mathrm{i}^3=\mathrm{i}^7=\mathrm{i}^{11}=\cdots=\mathrm{i}^{4N+3}=-\mathrm{i}$
4. $\mathrm{i}^4=\mathrm{i}^8=\mathrm{i}^{12}=\cdots=\mathrm{i}^{4N}=1$

Properties of Argand diagram:

1. Real axis: Horizontal axis
2. Imaginary axis: Vertical axis
3. $r=|z|=\sqrt{a^2+b^2}$: Modulus of $z=a+b\mathrm{i}$
4. $\theta=\arg(z)=\arctan\dfrac{b}{a}$: Argument of $z=a+b\mathrm{i}$



Forms of complex numbers:

1. $z=a+b\mathrm{i}$: Cartesian form
2. $z=r(\cos\theta+\mathrm{i}\sin\theta)=r\,\mathrm{cis}\,\theta$: Modulus-argument (polar) form
3. $z=re^{\mathrm{i}\theta}$: Euler (exponential) form

Properties of moduli and arguments of complex numbers z_1 and z_2:

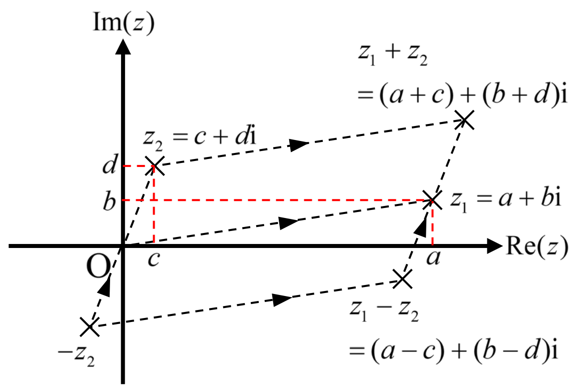
1. $|z_1 z_2| = |z_1||z_2|$
2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
3. $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
4. $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
5. $\arg(z_1^n) = n \arg z_1$

Applications to polynomials and useful expressions:

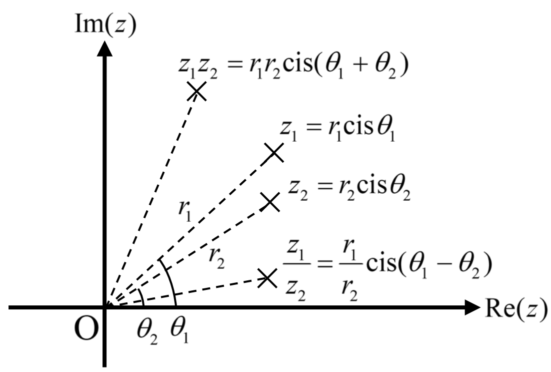
1. If $z = a + bi$ is a root of the polynomial equation $p(z) = 0$, then $z^* = a - bi$ is also a root of $p(z) = 0$
2. $zz^* = (a + b\mathrm{i})(a - b\mathrm{i}) = a^2 + b^2$

Geometric interpretation of complex numbers:

1. $(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}$
2. $(a + b\mathrm{i}) - (c + d\mathrm{i}) = (a - c) + (b - d)\mathrm{i}$



3. $(r_1 \mathrm{cis}\theta_1) \times (r_2 \mathrm{cis}\theta_2) = r_1 \times r_2 \ \mathrm{cis}(\theta_1 + \theta_2)$
4. $(r_1 \mathrm{cis}\theta_1) \div (r_2 \mathrm{cis}\theta_2) = (r_1 \div r_2) \ \mathrm{cis}(\theta_1 - \theta_2)$



Relationship between the sinusoidal models $y = A_1 \sin(Bx) + D_1$ and $y = A_2 \sin(Bx - C) + D_2$, A_1, A_2, B, C, D_1, $D_2 \in \mathbb{R}$:

1. C: Phase shift
2. Two sinusoidal models of the same frequency can be combined by considering their Euler forms

Notes on GDC		
TEXAS TI-84 Plus CE	**TEXAS TI-Nspire CX**	**CASIO fx-CG50**
mode→re^(θi) on the 8th row to express a complex number in its Euler form to find its modulus and argument →a+bi on the 8th row to express a complex number in its Cartesian form to find its real part and imaginary part	menu→Number →Complex Number Tools →Convert to Polar to express a complex number in its Euler form to find its modulus and argument →Convert to Rectangular to express a complex number in its Cartesian form to find its real part and imaginary part	SHIFT→MENU→F3 on the row Complex Mode to express a complex number in its Euler form to find its modulus and argument →F2 on the row Complex Mode to express a complex number in its Cartesian form to find its real part and imaginary part

Example 1.8

Consider the complex numbers $z_1 = 3\text{cis}\frac{\pi}{12}$ and $z_2 = 2\text{cis}\frac{\pi}{4}$.

(a) (i) Express $z_1^{\ 3}$ in the form $r\text{cis}\theta$.

[2]

$z_1^{\ 3}$

$= \left(3\text{cis}\frac{\pi}{12}\right)^3$

$= 27\text{cis}\frac{\pi}{4}$ $r = 27$ (A1) & $\theta = \frac{\pi}{4}$ (A1)

(ii) Hence, write down the real part of $z_1^{\ 3}$.

[1]

The real part of $z_1^{\ 3}$

$= 27\cos\frac{\pi}{4}$

$= 19.09188309$

$= 19.1$ (A1)

(b) Express $z_1^{\ 3}z_2$ in the form

(i) $r\text{cis}\theta$;

[2]

$z_1^{\ 3}z_2$

$= \left(27\text{cis}\frac{\pi}{4}\right)\left(2\text{cis}\frac{\pi}{4}\right)$

$= 54\text{cis}\frac{\pi}{2}$ $r = 54$ (A1) & $\theta = \frac{\pi}{2}$ (A1)

(ii) $re^{i\theta}$.

[1]

$z_1^{\ 3}z_2 = 54e^{\frac{\pi}{2}i}$ (A1)

(c) Find the least value of n, $n \in \mathbb{Z}^+$ such that $\left(\frac{z_2}{z_1}\right)^n \in \mathbb{R}$.

[3]

$\arg\left(\frac{z_2}{z_1}\right)^n$

$= n\arg\left(\frac{z_2}{z_1}\right)$

$= n\left(\frac{\pi}{4} - \frac{\pi}{12}\right)$ $\arg\left(\frac{z_2}{z_1}\right) = \arg z_2 - \arg z_1$ (M1)

$= \frac{n\pi}{6}$

$\therefore \frac{n\pi}{6} = \pi, 2\pi, 3\pi, \cdots$ Multiples of π (M1)

$\frac{n}{6} = 1, 2, 3, \cdots$

$n = 6, 12, 18, \cdots$

Thus, the least value of n is 6. (A1)

Exercise 1.8

Consider the complex numbers $z_1 = 2\text{cis}\frac{\pi}{8}$ and $z_2 = \text{cis}\frac{\pi}{6}$.

(a) (i) Express $z_1^{\ 4}$ in the form $r\text{cis}\theta$.

[2]

(ii) Hence, write down the imaginary part of $z_1^{\ 4}$.

[1]

(b) Express $\frac{z_2}{z_1^{\ 4}}$ in the form

(i) $r\text{cis}\theta$;

[2]

(ii) $re^{i\theta}$.

[1]

(c) Find the least value of n, $n \in \mathbb{Z}^+$ such that $\left(\frac{z_2}{z_1}\right)^n$ is purely imaginary.

[3]

Example 1.9

Two alternating current electrical sources with the same frequencies are combined, and the voltages from these sources can be expressed as $V_1 = 3\sin(bt)$ and $V_2 = 4\sin\left(bt - \frac{\pi}{2}\right)$, where b is the frequency. The combined total voltage can be expressed in the form $V_1 + V_2 = V_0 \sin(bt + \alpha)$. Find the values of V_0 and α.

[5]

V_1

$= 3\sin(bt)$

$= \mathrm{Im}(3e^{bt\mathrm{i}})$ — $\mathrm{Im}(z)$ (M1)

V_2

$= 4\sin\left(bt - \frac{\pi}{2}\right)$

$= \mathrm{Im}\left(4e^{\left(bt - \frac{\pi}{2}\right)\mathrm{i}}\right)$

$V_1 + V_2$

$= \mathrm{Im}(3e^{bt\mathrm{i}}) + \mathrm{Im}\left(4e^{\left(bt - \frac{\pi}{2}\right)\mathrm{i}}\right)$

$= \mathrm{Im}\left(3e^{bt\mathrm{i}} + 4e^{\left(bt - \frac{\pi}{2}\right)\mathrm{i}}\right)$ — $\mathrm{Im}(z_1) + \mathrm{Im}(z_2) = \mathrm{Im}(z_1 + z_2)$ (M1)

$= \mathrm{Im}\left(e^{bt\mathrm{i}}\left(3 + 4e^{-\frac{\pi}{2}\mathrm{i}}\right)\right)$

$= \mathrm{Im}\left(e^{bt\mathrm{i}}(5e^{-0.927295218\mathrm{i}})\right)$

$= \mathrm{Im}(5e^{bt\mathrm{i} - 0.927295218\mathrm{i}})$ — $5e^{bt\mathrm{i} - 0.927295218\mathrm{i}}$ (A1)

$= \mathrm{Im}(5e^{(bt - 0.927295218)\mathrm{i}})$

$= 5\sin(bt - 0.927295218)$

$\therefore V_0 = 5$ & $\alpha = -0.927$ — (A1)(A1)

Exercise 1.9

Two alternating current electrical sources with the same frequencies are combined, and the voltages from these sources can be expressed as $V_1 = 12\cos(bt)$ and $V_2 = 5\cos\left(bt + \frac{\pi}{16}\right)$, where b is the frequency. The combined total voltage can be expressed in the form $V_1 + V_2 = V_0\cos(bt + \alpha)$. Find the values of V_0 and α.

[5]

8 Matrices

Important Notes

Terminologies of matrices:

1. $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$: A $m\times n$ matrix with m rows and n columns
2. a_{ij}: Element on the i th row and the j th column
3. $\mathbf{I}=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$: Identity matrix
4. $\mathbf{0}=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$: Zero matrix
5. $\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$: Diagonal matrix
6. $|\mathbf{A}|=\det(\mathbf{A})$: Determinant of $\mathbf{A}$
7. $\mathbf{A}^{-1}$: Inverse of $\mathbf{A}$

Properties of $\mathbf{A}^{-1}$:

1. $\mathbf{A}$ is non-singular if $\det(\mathbf{A})\neq 0$
2. $\mathbf{A}^{-1}$ exists if $\mathbf{A}$ is non-singular

Properties of any 2×2 square matrices $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

1. $|\mathbf{A}|=\det(\mathbf{A})=ad-bc$: Determinant of $\mathbf{A}$
2. $\mathbf{A}^{-1}=\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$: Inverse of $\mathbf{A}$

Operations of matrices:

1. $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \pm \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} \pm b_{11} & \cdots & a_{1n} \pm b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & \cdots & a_{mn} \pm b_{mn} \end{pmatrix}$
2. $k\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & \ddots & \vdots \\ ka_{m1} & \cdots & ka_{mn} \end{pmatrix}$
3. $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$: The element on the ith row and the jth column of $\mathbf{C} = \mathbf{AB}$ $= \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nk} \end{pmatrix}$, where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are $m \times n$, $n \times k$ and $m \times k$ matrices respectively

Systems of equations:

1. A 2×2 system $\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$ can be expressed as $\mathbf{AX} = \mathbf{B}$, where $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ can be solved by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ $= \begin{pmatrix} a & b \\ d & e \end{pmatrix}^{-1}\begin{pmatrix} c \\ f \end{pmatrix}$
2. A 3×3 system $\begin{cases} ax + by + cz = d \\ ex + fy + gz = h \\ ix + jy + kz = l \end{cases}$ can be expressed as $\mathbf{AX} = \mathbf{B}$, where $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ can be solved by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ $= \begin{pmatrix} a & b & c \\ e & f & g \\ i & j & k \end{pmatrix}^{-1}\begin{pmatrix} d \\ h \\ l \end{pmatrix}$

Eigenvalues and eigenvectors of $\mathbf{A}$:

1. $\det(\mathbf{A} - \lambda\mathbf{I})$: Characteristic polynomial of $\mathbf{A}$
2. Solution(s) of $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$: Eigenvalue(s) of $\mathbf{A}$
3. $\mathbf{v}$: Eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, which satisfies $\mathbf{Av} = \lambda\mathbf{v}$

Diagonalization of $\mathbf{A}$:

1. $\mathbf{D}=\begin{pmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{pmatrix}$: Diagonal matrix of the eigenvalues of $\mathbf{A}$
2. $\mathbf{V}=\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{pmatrix}$: A matrix of the eigenvectors of $\mathbf{A}$ having the same order as the corresponding eigenvalues
3. $\mathbf{A}=\mathbf{V}\mathbf{D}\mathbf{V}^{-1} \Rightarrow \mathbf{A}^n=\mathbf{V}\mathbf{D}^n\mathbf{V}^{-1}$

Two-dimensional transformation matrices:

1. $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$: Reflection about the x-axis
2. $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$: Reflection about the y-axis
3. $\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}$: Reflection about the line $y=mx$, where $m=\tan\theta$
4. $\begin{pmatrix}1 & 0\\ 0 & k\end{pmatrix}$: Vertical stretch with scale factor k
5. $\begin{pmatrix}k & 0\\ 0 & 1\end{pmatrix}$: Horizontal stretch with scale factor k
6. $\begin{pmatrix}k & 0\\ 0 & k\end{pmatrix}$: Enlargement about the origin with scale factor k
7. $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$: Rotation with positive angle θ anticlockwise about the origin
8. $\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$: Rotation with positive angle θ clockwise about the origin
9. Area of the image $=\left|\det(\mathbf{T})\right|\times$ Area of the object, where $\mathbf{T}$ is the transformation matrix
10. $\mathbf{T}=\mathbf{B}\mathbf{A}$ represents the transformation matrix of the transformation by $\mathbf{A}$, followed by the transformation by $\mathbf{B}$

Notes on GDC		
TEXAS TI-84 Plus CE 2nd→x^{-1}→EDIT to input a 2×2 coefficient matrix and an identity matrix in [A] and [I] respectively y= to sketch the graph of $y=\det(A-xI)$ and the eigenvalues of the coefficient matrix $\mathbf{A}$ are the x-intercepts of the graph apps→PlySmlt2 →SIMULTANEOUS EQN SOLVER →Choose 2 in EQUATIONS and UNKNOWNS →For each eigenvalue, input the entries of the coefficient matrix $\mathbf{A}$ on the left hand side, where the first entry and the last entry are subtracted by the corresponding eigenvalue, and input two 0s on the right hand side →SOLVE to find the ratio of x to y, and use simple integers of x or y to find the corresponding eigenvector	**TEXAS TI-Nspire CX** templates→Matrix to input the coefficient matrix menu→Matrix & Vector →Advanced→Eigenvalues to find the eigenvalues of the coefficient matrix menu→Matrix & Vector →Advanced→Eigenvectors to find the eigenvectors of the coefficient matrix, with the magnitude of each eigenvector equals to 1	**CASIO fx-CG50** F3 to input a 2×2 coefficient matrix and an identity matrix in [A] and [I] respectively SHIFT→2 to define matrices Graph to sketch the graph of $y=\det(A-xI)$ and the eigenvalues of the coefficient matrix $\mathbf{A}$ are the x-intercepts of the graph Equation→F1→Choose 2 in Number Of Unknowns →For each eigenvalue, input the entries of the coefficient matrix $\mathbf{A}$ on the left hand side, where the first entry and the last entry are subtracted by the corresponding eigenvalue, and input two 0s on the right hand side →SOLVE to find the ratio of x to y, and use simple integers of x or y to find the corresponding eigenvector

Example 1.10

The matrix $\mathbf{A}$ is defined by $\mathbf{A}=\begin{pmatrix}2 & -1\\ 5 & -4\end{pmatrix}$. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{A}$, where $\lambda_1<\lambda_2$.

(a) Find the characteristic polynomial of $\mathbf{A}$.

[2]

The characteristic polynomial

$=\det(\mathbf{A}-\lambda\mathbf{I})$

$=\begin{vmatrix}2-\lambda & -1\\ 5 & -4-\lambda\end{vmatrix}$ — $\det(\mathbf{A}-\lambda\mathbf{I})$ (M1)

$=(2-\lambda)(-4-\lambda)-(-1)(5)$

$=-8-2\lambda+4\lambda+\lambda^2+5$

$=\lambda^2+2\lambda-3$ — (A1)

(b) (i) Write down the values of λ_1 and λ_2.

[2]

$\lambda_1=-3$, $\lambda_2=1$ — (A1)(A1)

(ii) Hence, write down $\mathbf{v}_1$ and $\mathbf{v}_2$, the eigenvectors of $\mathbf{A}$ corresponding to λ_1 and λ_2 respectively.

[2]

$\mathbf{v}_1=\begin{pmatrix}1\\5\end{pmatrix}$, $\mathbf{v}_2=\begin{pmatrix}1\\1\end{pmatrix}$ — (A1)(A1)

It is given that $\mathbf{A}^n=\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{P}$ is a 2×2 matrix and $\mathbf{D}$ is a 2×2 diagonal matrix.

(c) Write down

(i) $\mathbf{P}$;

[1]

$\mathbf{P}=\begin{pmatrix}1 & 1\\ 5 & 1\end{pmatrix}$ — (A1)

(ii) $\mathbf{D}^n$.

[2]

$\mathbf{D}^n$

$= \begin{pmatrix} (-3)^n & 0 \\ 0 & 1^n \end{pmatrix}$ $\begin{pmatrix} \lambda_1^{\ n} & 0 \\ 0 & \lambda_2^{\ n} \end{pmatrix}$ (A1)

$= \begin{pmatrix} (-3)^n & 0 \\ 0 & 1 \end{pmatrix}$ (A1)

(d) Hence, express $\mathbf{A}^n$ in terms of n.

[3]

$\mathbf{A}^n$

$= \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$

$= \begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} (-3)^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix}^{-1}$

$= \begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} (-3)^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{5}{4} & -\frac{1}{4} \end{pmatrix}$ $\begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{5}{4} & -\frac{1}{4} \end{pmatrix}$ (A1)

$= \begin{pmatrix} (-3)^n & 1 \\ 5(-3)^n & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{5}{4} & -\frac{1}{4} \end{pmatrix}$ $\begin{pmatrix} (-3)^n & 1 \\ 5(-3)^n & 1 \end{pmatrix}$ (A1)

$= \begin{pmatrix} -\frac{1}{4}(-3)^n + \frac{5}{4} & \frac{1}{4}(-3)^n - \frac{1}{4} \\ -\frac{5}{4}(-3)^n + \frac{5}{4} & \frac{5}{4}(-3)^n - \frac{1}{4} \end{pmatrix}$ (A1)

Exercise 1.10

The matrix $\mathbf{A}$ is defined by $\mathbf{A} = \begin{pmatrix} 3 & -12 \\ -2 & 5 \end{pmatrix}$. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{A}$, where $\lambda_1 < \lambda_2$.

(a) Find the characteristic polynomial of $\mathbf{A}$.

[2]

(b) (i) Write down the values of λ_1 and λ_2.

[2]

(ii) Hence, write down $\mathbf{v}_1$ and $\mathbf{v}_2$, the eigenvectors of $\mathbf{A}$ corresponding to λ_1 and λ_2 respectively.

[2]

It is given that $\mathbf{A}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{P}$ is a 2×2 matrix and $\mathbf{D}$ is a 2×2 diagonal matrix.

(c) Write down

(i) $\mathbf{P}$;

[1]

(ii) $\mathbf{D}^n$.

[1]

(d) Hence, express $\mathbf{A}^n$ in terms of n.

[3]

Example 1.11

A quadcopter is programmed to complete a series of movements in a horizontal plane relative to an origin O and a set of coordinate axes. Below are the steps of transformations in order, and the corresponding transformation matrix:

1. a clockwise rotation of $\frac{\pi}{6}$ radians about O, represented by $\mathbf{A}$
2. a reflection in the line $y=-\sqrt{3}x$, represented by $\mathbf{B}$
3. an anticlockwise rotation of $\frac{\pi}{3}$ radians about O, represented by $\mathbf{C}$

(a) Find

(i) $\mathbf{A}$;

[2]

$$\mathbf{A} = \begin{pmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix}$$

$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ (M1)

$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

(A1)

(ii) $\mathbf{B}$;

[3]

$$\tan\theta = -\sqrt{3}$$

$$\theta = \frac{2\pi}{3}$$

$\theta = \frac{2\pi}{3}$ (A1)

$$\mathbf{B} = \begin{pmatrix} \cos 2\left(\frac{2\pi}{3}\right) & \sin 2\left(\frac{2\pi}{3}\right) \\ \sin 2\left(\frac{2\pi}{3}\right) & -\cos 2\left(\frac{2\pi}{3}\right) \end{pmatrix}$$

$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ (M1)

$$= \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

(A1)

(iii) $\mathbf{C}$;

[2]

$\mathbf{C}$

$= \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}$ $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ (M1)

$= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ (A1)

(iv) $\mathbf{T}$, the matrix defining the transformation that represents the overall change in position.

[2]

$\mathbf{T}$

$= \mathbf{CBA}$ $\mathbf{CBA}$ (M1)

$= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$ (A1)

(b) (i) Write down $\mathbf{T}^2$.

[1]

$\mathbf{T}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (A1)

(ii) Hence, state what $\mathbf{T}^2$ indicates for the possible movement.

[2]

$\mathbf{T}^2$ indicates that the three movements are repeated, and the quadcopter will (A1)
return to its original position. (A1)

The four rotors of the quadcopter are initially at P, Q, R and S, and they are positioned at P', Q', R' and S' respectively after performing the above movements.

(c) Find a single transformation that is equivalent to the three transformations represented by $\mathbf{T}$.

[4]

$$\mathbf{T} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$ $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ (M1)

$$\therefore \begin{cases} \sin 2\theta = -\frac{1}{2} \\ \cos 2\theta = \frac{\sqrt{3}}{2} \end{cases}$$ System of equations (M1)

$$\tan 2\theta = -\frac{1}{\sqrt{3}}$$ $\tan 2\theta = -\frac{1}{\sqrt{3}}$ (A1)

$$2\theta = \frac{5\pi}{6}$$

$$\theta = \frac{5\pi}{12}$$

Thus, the required single transformation is a

reflection in the line $y = \left(\tan \frac{5\pi}{12}\right)x$. (A1)

(d) Someone claims that the area of the quadrilateral PQRS is the same as the area of the quadrilateral $\mathrm{P'Q'R'S'}$. Is the claim correct? Explain your answer.

[3]

$|\det(\mathbf{T})|$

$= \left\| \begin{matrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{matrix} \right\|$

$= \left| \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) \right|$

$= |-1|$

$= 1$ $|\det(\mathbf{T})| = 1$ (A1)

Area of the quadrilateral $\mathrm{P'Q'R'S'}$

$= |\det(\mathbf{T})| \times$ Area of the quadrilateral PQRS

$=$ Area of the quadrilateral PQRS Area ratio (R1)

Thus, the claim is correct. (A1)

Exercise 1.11

A toy with three wheels is controlled to complete a series of movements in a horizontal plane relative to an origin O and a set of coordinate axes. Below are the steps of transformations in order, and the corresponding transformation matrix:

1. a clockwise rotation of $\frac{\pi}{3}$ radians about O, represented by $\mathbf{A}$
2. a reflection in the x-axis, represented by $\mathbf{B}$
3. a reflection in the line $y = x$, represented by $\mathbf{C}$

(a) Find

(i) $\mathbf{A}$;

[2]

(ii) $\mathbf{B}$;

[3]

(iii) $\mathbf{C}$;

[3]

(iv) $\mathbf{T}$, the matrix defining the transformation that represents the overall change in position.

[2]

(b) (i) Write down $\mathbf{T}^{12}$.

[1]

(ii) Hence, state what $\mathbf{T}^{12}$ indicates for the possible movement.

[2]

The three wheels of the toy are initially at U, V and W, and they are positioned at U', V' and W' respectively after performing the above movements.

(c) Find a single transformation that is equivalent to the three transformations represented by $\mathbf{T}$.

[4]

(d) Someone claims that the area of the triangle UVW is the same as the area of the triangle $\mathrm{U'V'W'}$. Is the claim correct? Explain your answer.

[3]