Formula List of
Analysis and Approaches
Higher Level
for IBDP Mathematics

	Intensive Notes by Topics here		IBDP Maths Info. & Exam Tricks here
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1 Standard Form

✓ Standard Form:

A number in the form $(\pm)a\times10^{k}$, where $1\le a<10$ and k is an integer

2 Quadratic Functions

✓ General form $y=ax^2+bx+c$, where $a\neq0$:

$a>0$	The graph opens upward
$a<0$	The graph opens downward
c	y-intercept
$h=-\frac{b}{2a}$	x-coordinate of the vertex
$k=ah^2+bh+c$	y-coordinate of the vertex
	Extreme value of y
$x=h$	Equation of the axis of symmetry

✓ Other forms:

1. $y=a(x-h)^2+k$: Vertex form
2. $y=a(x-p)(x-q)$: Factored form with x-intercepts p and q

✓ Solving the quadratic equation $ax^2+bx+c=0$, where $a\neq0$:

1. Factorization by cross method
2. $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$: Quadratic Formula
3. Method of completing the square

✓ The discriminant $\Delta = b^2 - 4ac$ of $ax^2 + bx + c = 0$:

$\Delta > 0$	The quadratic equation has two distinct real roots
$\Delta = 0$	The quadratic equation has one double real root
$\Delta < 0$	The quadratic equation has no real root

✓ The x-intercepts of the quadratic function $y = ax^2 + bx + c$ are the roots of the corresponding quadratic equation $ax^2 + bx + c = 0$

3 Functions

✓ The function $y = f(x)$:

1. $f(a)$: Functional value when $x = a$
2. Set of values of x: Domain
3. Set of values of y: Range

✓ $f \circ g(x) = f(g(x))$: Composite function when $g(x)$ is substituted into $f(x)$

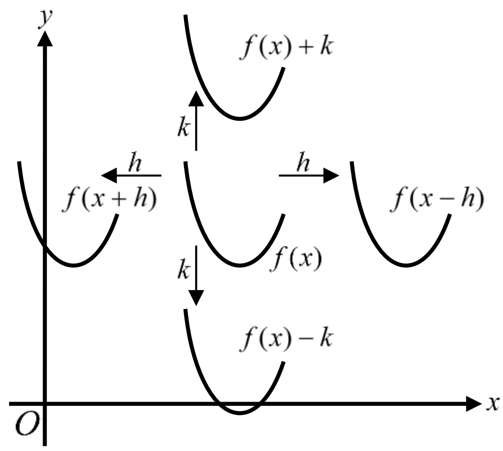
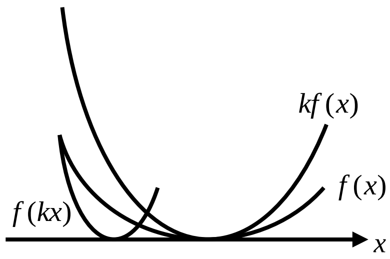
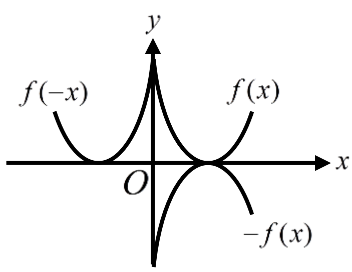
✓ Steps of finding the inverse function $y = f^{-1}(x)$ of $f(x)$:

1. Start from expressing y in terms of x
2. Interchange x and y
3. Make y the subject in terms of x

✓ Properties of $y = f^{-1}(x)$:

1. $f(f^{-1}(x)) = f^{-1}(f(x)) = x$
2. The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about $y = x$

✓ Summary of transformations:

	$f(x)\to f(x)+k$: Translate upward by k units
	$f(x)\to f(x)-k$: Translate downward by k units
	$f(x)\to f(x+h)$: Translate to the left by h units
	$f(x)\to f(x-h)$: Translate to the right by h units
	$f(x)\to kf(x)$: Vertical stretch of scale factor k
	$f(x)\to f(kx)$: Horizontal compression of scale factor k
	$f(x)\to -f(x)$: Reflection about the x-axis
	$f(x)\to f(-x)$: Reflection about the y-axis

✓ Properties of rational function $y=\dfrac{ax+b}{cx+d}$:

1. $y=\dfrac{1}{x}$: Reciprocal function

2. $y=\dfrac{a}{c}$: Horizontal asymptote

3. $x=-\dfrac{d}{c}$: Vertical asymptote

✓ Odd and even functions:

$f(x)$ is odd if $f(-x)=-f(x)$

$f(x)$ is even if $f(-x)=f(x)$

✓ $f^{-1}(x)$ exists only when $f(x)$ is one-to-one in the restricted domain

✓ Absolute function:

$$|f(x)| = \begin{cases} f(x) \text{ if } x \geq 0 \\ -f(x) \text{ if } x < 0 \end{cases}$$

4 Exponential and Logarithmic Functions

✓ $y = a^x$: Exponential function of base $a \neq 1$

✓ Methods of solving an exponential equation $a^x = b$:

1. Change b into a^y such that $a^x = a^y \Rightarrow x = y$
2. Take logarithm for both sides

✓ $y = \log_a x$: Logarithmic function of base $a > 0$

✓ $y = \log x = \log_{10} x$: Common Logarithmic function

✓ $y = \ln x = \log_e x$: Natural Logarithmic function, where $e = 2.71828...$ is an exponential number

✓ Laws of logarithm, where a, b, c, p, q, $x > 0$:

1. $x = a^y \Leftrightarrow y = \log_a x$
2. $\log_a 1 = 0$
3. $\log_a a = 1$
4. $\log_a p + \log_a q = \log_a pq$
5. $\log_a p - \log_a q = \log_a \dfrac{p}{q}$
6. $\log_a p^n = n \log_a p$
7. $\log_b a = \dfrac{\log_c a}{\log_c b}$

✓ Properties of the graphs of $y = a^x$:

$a > 1$	$0 < a < 1$
y-intercept $= 1$	
y increases as x increases	y decreases as x increases
y tends to zero as x tends to negative infinity	y tends to zero as x tends to positive infinity
Horizontal asymptote: $y = 0$	

✓ Properties of the graphs of $y = \log_a x$:

$a > 1$	$0 < a < 1$
x-intercept $= 1$	
y increases as x increases	y decreases as x increases
x tends to zero as y tends to negative infinity	x tends to zero as y tends to positive infinity
Vertical asymptote: $x = 0$	

5 Polynomials

✓ Number of roots of the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$:
The maximum number of roots of $f(x) = 0$ is n

✓ Sum and product of roots of $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$:

1. r_1, r_2, ..., r_n: Roots
2. $r_1 + r_2 + \cdots + r_n = -\dfrac{a_{n-1}}{a_n}$
3. $r_1 r_2 r_3 \cdots r_{n-1} r_n = (-1)^n \dfrac{a_0}{a_n}$

✓ Factor theorem:
$(x - a)$ is a factor of $f(x)$ if $f(a) = 0$

$(px - q)$ is a factor of $f(x)$ if $f\left(\dfrac{q}{p}\right) = 0$

✓ Remainder theorem:

$f(a)$ is the remainder when $f(x)$ is divided by $(x-a)$

$f\left(\frac{q}{p}\right)$ is the remainder when $f(x)$ is divided by $(px-q)$

✓ Partial fractions:

1. $\frac{ax+b}{(cx+d)(ex+f)}$ can be expressed as $\frac{P}{cx+d}+\frac{Q}{ex+f}$
2. $\frac{ax+b}{(cx+d)^2}$ can be expressed as $\frac{P}{cx+d}+\frac{Q}{(cx+d)^2}$

6 Systems of Equations

✓ $\begin{cases} ax+by=c \\ dx+ey=f \end{cases}$: 2×2 system

✓ $\begin{cases} ax+by+cz=d \\ ex+fy+gz=h \\ ix+jy+kz=l \end{cases}$: 3×3 system

✓ The above systems can be solved by PlySmlt2 in TI-84 Plus CE

✓ Row operations of a system with row R_i:

1. Multiply the constant k to the row R_i (kR_i)
2. Add the row R_i to the row R_j (R_i+R_j)
3. Add the multiple of the row R_i to the row R_j (kR_i+R_j)

✓ Number of solutions of a system with the last row $az=b$ after row operation:

1. The system has a unique solution if $a\neq0$
2. The system has no solution if $a=0$ and $b\neq0$
3. The system has infinitely number of solutions if $a=0$ and $b=0$

7 Arithmetic Sequences

✓ Properties of an arithmetic sequence u_n:

1. u_1: First term
2. $d = u_2 - u_1 = u_n - u_{n-1}$: Common difference
3. $u_n = u_1 + (n-1)d$: General term (nth term)
4. $S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right] = \frac{n}{2}\left[u_1 + u_n\right]$: The sum of the first n terms

✓ $\sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + \cdots + u_{n-1} + u_n$: Summation sign

8 Geometric Sequences

✓ Properties of a geometric sequence u_n:

1. u_1: First term
2. $r = u_2 \div u_1 = u_n \div u_{n-1}$: Common ratio
3. $u_n = u_1 \times r^{n-1}$: General term (nth term)
4. $S_n = \frac{u_1(1-r^n)}{1-r}$: The sum of the first n terms
5. $S_\infty = \frac{u_1}{1-r}$: The sum to infinity, given that $-1 < r < 1$

9 Binomial Theorem

✓ Properties of the n factorial $n!$:

1. $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$
2. $0! = 1$
3. $n! = n \times (n-1)!$

✓ Properties of the combination coefficient $\binom{n}{r}$:

1. $\binom{n}{r}=\frac{n!}{r!(n-r)!}$
2. $\binom{n}{0}=\binom{n}{n}=1$
3. $\binom{n}{1}=\binom{n}{n-1}=n$
4. $\binom{n}{r}=\binom{n}{n-r}=\frac{n(n-1)\cdots(n-r+1)}{r!}$

✓ The binomial theorem:

$$(a+b)^n=\binom{n}{0}a^nb^0+\binom{n}{1}a^{n-1}b^1+\binom{n}{2}a^{n-2}b^2+\cdots+\binom{n}{n-1}a^1b^{n-1}+\binom{n}{n}a^0b^n$$

$=\sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$, where the $(r+1)$-th term $=\binom{n}{r}a^{n-r}b^r$

✓ Extended binomial theorem for $|x|<1$:

$$(1+x)^n=1+nx+\binom{n}{2}x^2+\binom{n}{3}x^3+\cdots$$

$$(1+x)^n=1+nx+\frac{(n)(n-1)}{(2)(1)}x^2+\frac{(n)(n-1)(n-2)}{(3)(2)(1)}x^3+\cdots$$

10 Mathematical Induction

✓ Steps of proving by mathematical induction:

1. Prove that the statement $P(n)$ is true when $n=1$
2. Assume that $P(n)$ is true when $n=k$
3. Prove that the statement $P(n)$ is true when $n=k+1$
4. Conclude that $P(n)$ is true for all positive integer n

✓ Types of mathematical induction:

1. General case
2. Divisibility

11 Proofs and Identities

✓ Identity of x: The equivalence of two expressions for all values of x

$\equiv$: Identity sign

✓ Types of proofs:

1. Prove by contradiction
2. Prove by counter example

12 Coordinate Geometry

✓ Consider the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on a x - y plane:

1. $m = \frac{y_2 - y_1}{x_2 - x_1}$: Slope of PQ
2. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$: Distance between P and Q
3. $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$: The mid-point of PQ

✓ Forms of straight lines with slope m and y-intercept c:

1. $y = mx + c$: Slope-intercept form
2. $Ax + By + C = 0$: General form

✓ Ways to find the x-intercept and the y-intercept of a line:

1. Substitute $y = 0$ and make x the subject to find the x-intercept
2. Substitute $x = 0$ and make y the subject to find the y-intercept

13 Trigonometry

✓ Trigonometric identities:

1. $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$

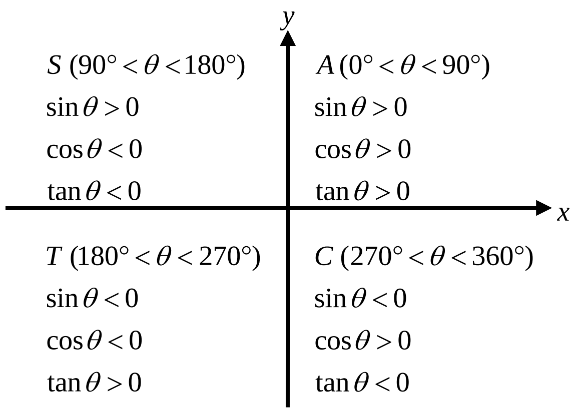
2. $\sin^2\theta + \cos^2\theta \equiv 1$

✓ Double angle formula:

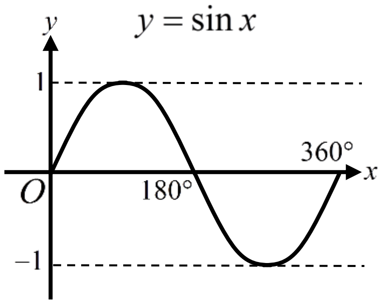
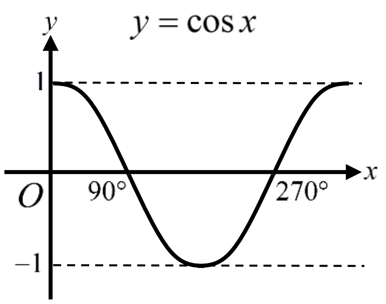
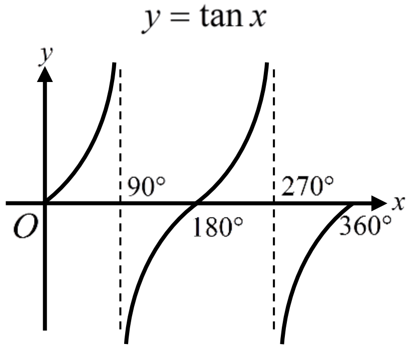
1. $\sin 2\theta = 2\sin\theta\cos\theta$

2. $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$

✓ ASTC diagram



✓ Properties of graphs of trigonometric functions:

	1. Amplitude $=1$ 2. Period $=360°$ 3. $-1\le \sin x\le 1$
	1. Amplitude $=1$ 2. Period $=360°$ 3. $-1\le \cos x\le 1$
	1. Period $=180°$ 2. $\tan x\in\mathbb{R}$ 3. Vertical asymptotes: $x=90°$, $x=270°$

✓ Properties of a general trigonometric function $y=A\sin B(x-C)+D$:

1. $A=\dfrac{y_{\max}-y_{\min}}{2}$: Amplitude
2. $B=\dfrac{2\pi}{\text{Period}}$
3. $D=\dfrac{y_{\max}+y_{\min}}{2}$
4. C can be found by substitution of a point on the graph

✓ Reciprocal trigonometric ratios:

1. $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$
2. $\sec\theta = \dfrac{1}{\cos\theta}$
3. $\cot\theta = \dfrac{1}{\tan\theta}$

✓ Inverse trigonometric functions:

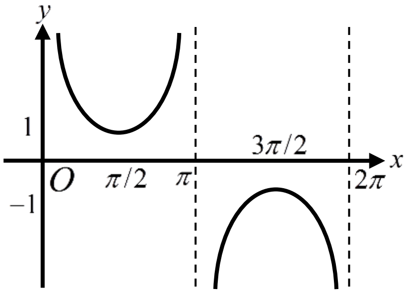
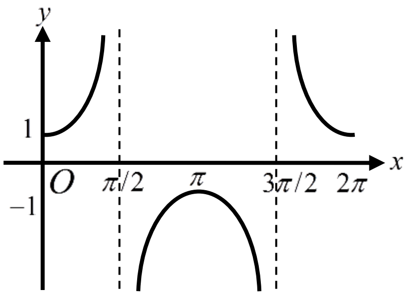
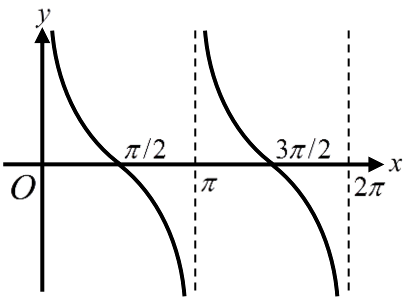
1. $f(x) = \sin x \Rightarrow f^{-1}(x) = \sin^{-1} x = \arcsin x$
2. $g(x) = \cos x \Rightarrow g^{-1}(x) = \cos^{-1} x = \arccos x$
3. $h(x) = \tan x \Rightarrow h^{-1}(x) = \tan^{-1} x = \arctan x$

✓ More trigonometric identities:

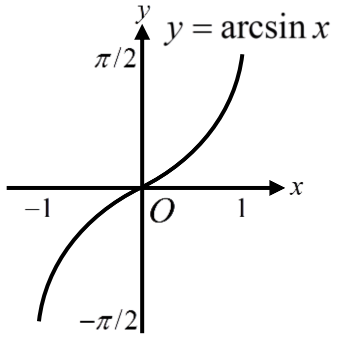
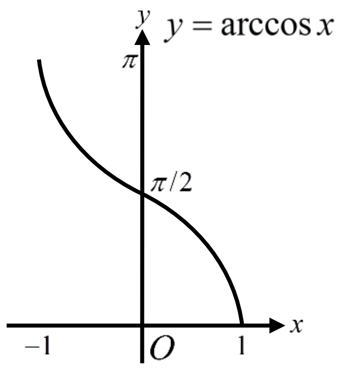
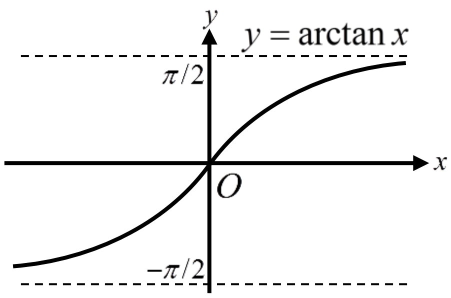
1. $\sec^2\theta = 1 + \tan^2\theta$
2. $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

✓ $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$: Double angle formula for tangent ratio

✓ Properties of graphs of reciprocal trigonometric functions:

$y = \operatorname{cosec} x$ 	1. Period $= 2\pi$ 2. $\operatorname{cosec} x \geq 1$ or $\operatorname{cosec} x \leq -1$ 3. Vertical asymptotes: $x = n\pi$, $n \in \mathbb{Z}$
$y = \sec x$ 	1. Period $= 2\pi$ 2. $\sec x \geq 1$ or $\sec x \leq -1$ 3. Vertical asymptotes: $x = n\pi + \frac{\pi}{2}$, $n \in \mathbb{Z}$
$y = \cot x$ 	1. Period $= \pi$ 2. $\cot x \in \mathbb{R}$ 3. Vertical asymptotes: $x = n\pi$, $n \in \mathbb{Z}$

✓ Properties of graphs of inverse trigonometric functions:

	1. $-1 \le x \le 1$ 2. $-\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}$
	1. $-1 \le x \le 1$ 2. $0 \le \arccos x \le \pi$
	1. $x \in \mathbb{R}$ 2. $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$

✓ Symmetric properties of trigonometric functions:

1. $\sin(-x) = -\sin x \Rightarrow \operatorname{cosec}(-x) = -\operatorname{cosec} x$
2. $\cos(-x) = \cos x \Rightarrow \sec(-x) = \sec x$
3. $\tan(-x) = -\tan x \Rightarrow \cot(-x) = -\cot x$

✓ Compound angle formula:

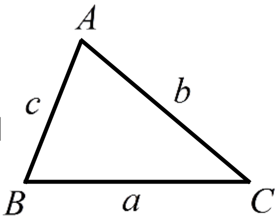
1. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
2. $\sin(A-B) = \sin A\cos B - \cos A\sin B$
3. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
4. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
5. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$
6. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

14 2-D Trigonometry

✓ Consider a triangle ABC:

1. $\frac{\sin A}{a}=\frac{\sin B}{b}$ or $\frac{a}{\sin A}=\frac{b}{\sin B}$: Sine rule

Note: The ambiguous case exists if two sides and an angle are known, and the angle is opposite to the shorter known side



2. $a^2=b^2+c^2-2bc\cos A$ or $\cos A=\frac{b^2+c^2-a^2}{2bc}$: Cosine rule

3. $\frac{1}{2}ab\sin C$: Area of the triangle ABC

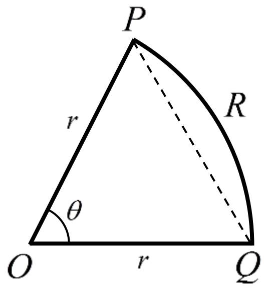
✓ $\frac{x^\circ}{180^\circ}=\frac{y\text{ rad}}{\pi\text{ rad}}$: Method of conversions between degree and radian

✓ Consider a sector $OPRQ$ with centre O, radius r and $\angle POQ=\theta$ in radian:

1. $r\theta$: Arc length PQ

2. $\frac{1}{2}r^2\theta$: Area of the sector $OPRQ$

3. $\frac{1}{2}r^2(\theta-\sin\theta)$: Area of the segment PRQ



15 Areas and Volumes

✓ For a cube of side length l:

1. $6l^2$: Total surface area

2. l^3: Volume

✓ For a cuboid of side lengths a, b and c:

1. $2(ab+bc+ac)$: Total surface area

2. abc: Volume

✓ For a prism of height h and cross-sectional area A:

1. Ah: Volume

✓ For a cylinder of height h and radius r:

1. $2\pi r^2 + 2\pi rh$: Total surface area
2. $2\pi rh$: Lateral surface area
3. $\pi r^2 h$: Volume

✓ For a pyramid of height h and base area A:

1. $\frac{1}{3}Ah$: Volume

✓ For a circular cone of height h and radius r:

1. $l = \sqrt{r^2 + h^2}$: Slant height
2. $\pi r^2 + \pi rl$: Total surface area
3. πrl : Curved surface area
4. $\frac{1}{3}\pi r^2 h$: Volume

✓ For a sphere of radius r:

1. $4\pi r^2$: Total surface area
2. $\frac{4}{3}\pi r^3$: Volume

✓ For a hemisphere of radius r:

1. $3\pi r^2$: Total surface area
2. $2\pi r^2$: Curved surface area
3. $\frac{2}{3}\pi r^3$: Volume

16 Vectors

✓ Terminologies of vectors:

$\overrightarrow{AB}$: Vector of length AB with initial point A and terminal point B

$\overrightarrow{OP}$: Position vector of P, where O is the origin

$\left|\overrightarrow{AB}\right|$: Magnitude (length) of $\overrightarrow{AB}$

$\hat{\mathbf{v}} = \frac{1}{|\mathbf{v}|}\mathbf{v}$: Unit vector parallel to $\mathbf{v}$, with $|\hat{\mathbf{v}}| = 1$

$\mathbf{0}$: Zero vector

$\mathbf{i}$: Unit vector along the positive x-axis

$\mathbf{j}$: Unit vector along the positive y-axis

$\mathbf{k}$: Unit vector along the positive z-axis

✓ A vector $\mathbf{v}$ can be expressed as $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ or $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

✓ Properties of vectors:

1. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
2. $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \pm \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 \pm v_1 \\ u_2 \pm v_2 \\ u_3 \pm v_3 \end{pmatrix}$
3. $\mathbf{v}$ and $k\mathbf{v}$ are in the same direction if $k > 0$
4. $\mathbf{v}$ and $k\mathbf{v}$ are in opposite direction if $k < 0$
5. $k\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} kv_1 \\ kv_2 \\ kv_3 \end{pmatrix}$

✓ Properties of the scalar product $\mathbf{u}\cdot\mathbf{v}$ of $\mathbf{u}=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$:

1. $\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2+u_3v_3=|\mathbf{u}||\mathbf{v}|\cos\theta$
2. $\mathbf{i}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{k}=1$
3. $\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0$
4. $\mathbf{u}$ and $\mathbf{v}$ are in the same direction if $\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}||\mathbf{v}|$
5. $\mathbf{u}$ and $\mathbf{v}$ are in opposite direction if $\mathbf{u}\cdot\mathbf{v}=-|\mathbf{u}||\mathbf{v}|$
6. $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if $\mathbf{u}\cdot\mathbf{v}=0$
7. $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$
8. $\mathbf{u}\cdot\mathbf{u}=|\mathbf{u}|^2$

✓ Properties of the vector product $\mathbf{u}\times\mathbf{v}$ of $\mathbf{u}=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$:

1. $\mathbf{u}\times\mathbf{v}=\begin{pmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{pmatrix}=|\mathbf{u}||\mathbf{v}|\sin\theta\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}//(\mathbf{u}\times\mathbf{v})$
2. $\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=\mathbf{0}$
3. $\mathbf{i}\times\mathbf{j}=\mathbf{k}$, $\mathbf{j}\times\mathbf{k}=\mathbf{i}$ and $\mathbf{k}\times\mathbf{i}=\mathbf{j}$
4. $\mathbf{j}\times\mathbf{i}=-\mathbf{k}$, $\mathbf{k}\times\mathbf{j}=-\mathbf{i}$ and $\mathbf{i}\times\mathbf{k}=-\mathbf{j}$
5. $\mathbf{u}$ and $\mathbf{v}$ are parallel if $\mathbf{u}\times\mathbf{v}=0$
6. $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if $|\mathbf{u}\times\mathbf{v}|=|\mathbf{u}||\mathbf{v}|$
7. $\mathbf{u}\times\mathbf{v}=-(\mathbf{v}\times\mathbf{u})$

✓ The area of the parallelogram with adjacent sides $\vec{AB}$ and $\vec{AD}$ is $\left|\vec{AB}\times\vec{AD}\right|$

✓ The area of the triangle with adjacent sides $\vec{AB}$ and $\vec{AD}$ is $\frac{1}{2}\left|\vec{AB}\times\vec{AD}\right|$

✓ The volume of the parallelepiped formed by $\vec{AB}$, $\vec{AD}$ and $\vec{AF}$ is $\left|\left(\vec{AB}\times\vec{AD}\right)\cdot\vec{AF}\right|$

✓ Forms of the straight line with fixed point $\mathrm{A}(a_1, a_2, a_3)$ and direction vector $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$:

1. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, $t \in \mathbb{R}$

2. $\begin{cases} x = a_1 + b_1 t \\ y = a_2 + b_2 t \\ z = a_3 + b_3 t \end{cases}$: Parametric form

3. $\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = \dfrac{z - a_3}{b_3} (= t)$: Cartesian equations

✓ Intersections of two lines:
1. Intersect at one point (One intersection)
2. Skew (No intersection)
3. Parallel (No intersection)
4. Coincide (Infinite number of intersections)

✓ Forms of the plane with fixed point $\mathrm{A}(a_1, a_2, a_3)$ and normal vector $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$:

1. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$

2. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda\mathbf{u} + \mu\mathbf{v}$, λ, $\mu \in \mathbb{R}$, where $\mathbf{u}$ and $\mathbf{v}$ are two non-parallel vectors on the plane

3. $n_1 x + n_2 y + n_3 z = a_1 n_1 + a_2 n_2 + a_3 n_3$: Cartesian form

✓ Intersections of two planes:
1. Intersect at one line
2. Parallel (No intersection)
3. Coincide (Infinite number of intersections)

17 Complex Numbers

✓ Terminologies of complex numbers:

$\mathrm{i}=\sqrt{-1}$: Imaginary unit

$z=a+b\mathrm{i}$: Complex number in Cartesian form

a : Real part of z

b : Imaginary part of z

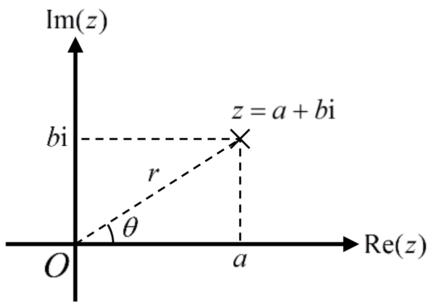
$z^*=a-b\mathrm{i}$: Conjugate of $z=a+b\mathrm{i}$

$|z|=r=\sqrt{a^2+b^2}$: Modulus of $z=a+b\mathrm{i}$

$\arg(z)=\theta=\arctan\frac{b}{a}$: Argument of $z=a+b\mathrm{i}$

✓ Properties of Argand diagram:

1. Real axis: Horizontal axis
2. Imaginary axis: Vertical axis



✓ Forms of complex numbers:

1. $z=a+b\mathrm{i}$: Cartesian form
2. $z=r(\cos\theta+\mathrm{i}\sin\theta)=r\operatorname{cis}\theta$: Modulus-argument form
3. $z=re^{\mathrm{i}\theta}$: Euler form

✓ Properties of moduli and arguments of complex numbers z_1 and z_2:

1. $|z_1z_2|=|z_1||z_2|$
2. $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$
3. $\arg(z_1z_2)=\arg z_1+\arg z_2$
4. $\arg\left(\frac{z_1}{z_2}\right)=\arg z_1-\arg z_2$

✓ If $z=a+b\mathrm{i}$ is a root of the polynomial equation $p(z)=0$, then $z^*=a-b\mathrm{i}$ is also a root of $p(z)=0$

✓ The roots of the equation $z^n = r\text{cis}\theta$ are $z = r^{\frac{1}{n}}\text{cis}\dfrac{\theta + 2k\pi}{n}$, $k = 0, 1, 2, \cdots, n-1$

✓ De Moivre's theorem:

If $z = r\text{cis}\theta$, then $z^n = r^n\text{cis}n\theta$

18 Differentiation

✓ Derivatives of a function $y = f(x)$:

1. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$: First derivative
2. $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right) = f''(x)$: Second derivative
3. $\dfrac{\mathrm{d}^ny}{\mathrm{d}x^n} = f^{(n)}(x)$: n-th derivative

✓ Rules of differentiation:

$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$	$f(x) = p(x) + q(x) \Rightarrow f'(x) = p'(x) + q'(x)$
$f(x) = \sin x \Rightarrow f'(x) = \cos x$	$f(x) = cp(x) \Rightarrow f'(x) = cp'(x)$
$f(x) = \cos x \Rightarrow f'(x) = -\sin x$	$f(x) = p(q(x)) \Rightarrow f'(x) = p'(q(x)) \cdot q'(x)$
$f(x) = \tan x \Rightarrow f'(x) = \dfrac{1}{\cos^2 x}$	$f(x) = p(x)q(x)$ $\Rightarrow f'(x) = p'(x)q(x) + p(x)q'(x)$
$f(x) = e^x \Rightarrow f'(x) = e^x$	$f(x) = \dfrac{p(x)}{q(x)}$ $\Rightarrow f'(x) = \dfrac{p'(x)q(x) - p(x)q'(x)}{(q(x))^2}$
$f(x) = \ln x \Rightarrow f'(x) = \dfrac{1}{x}$	

✓ Relationships between graph properties and the derivatives:

1. $f'(x) > 0$ for $a \le x \le b$: $f(x)$ is increasing in the interval
2. $f'(x) < 0$ for $a \le x \le b$: $f(x)$ is decreasing in the interval
3. $f'(a) = 0$: $(a,\ f(a))$ is a stationary point of $f(x)$
4. $f'(a) = 0$ and $f'(x)$ changes from positive to negative at $x = a$: $(a,\ f(a))$ is a maximum point of $f(x)$
5. $f'(a) = 0$ and $f'(x)$ changes from negative to positive at $x = a$: $(a,\ f(a))$ is a minimum point of $f(x)$
6. $f''(a) = 0$ and $f''(x)$ changes sign at $x = a$: $(a,\ f(a))$ is a point of inflexion of $f(x)$

✓ Slopes of tangents and normals:

1. $f'(a)$: Slope of tangent at $x = a$
2. $\dfrac{-1}{f'(a)}$: Slope of normal at $x = a$

✓ Differentiation by first principle:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

✓ More differentiation rules:

$f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$	$f(x) = \cot x \Rightarrow f'(x) = -\operatorname{cosec}^2 x$
$f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$	$f(x) = \operatorname{cosec} x \Rightarrow f'(x) = -\operatorname{cosec} x \cot x$
$f(x) = a^x \Rightarrow f'(x) = a^x \ln a$	$f(x) = \log_a x \Rightarrow f'(x) = \dfrac{1}{x \ln a}$
$f(x) = \arcsin x \Rightarrow f'(x) = \dfrac{1}{\sqrt{1-x^2}}$	$f(x) = \arccos x \Rightarrow f'(x) = -\dfrac{1}{\sqrt{1-x^2}}$
$f(x) = \arctan x \Rightarrow f'(x) = \dfrac{1}{1+x^2}$	

✓ Implicit differentiation:

$$F(x, y) = G(x, y) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x} F(x, y) = \frac{\mathrm{d}}{\mathrm{d}x} G(x, y)$$

19 Applications of Differentiation

✓ Equations of tangents and normals:

1. $y - f(a) = f'(a)(x-a)$: Equation of tangent at $x = a$
2. $y - f(a) = \left(\frac{-1}{f'(a)}\right)(x-a)$: Equation of normal at $x = a$

✓ $\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\mathrm{d}N}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t}$: Rate of change of N with respect to the time t

✓ Tests for optimization:

1. First derivative test
2. Second derivative test

✓ Applications in kinematics:

1. $s(t)$: Displacement with respect to the time t
2. $v(t) = s'(t)$: Velocity
3. $a(t) = v'(t)$: Acceleration

✓ Properties of rate of change:

1. $\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\mathrm{d}N}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t}$
2. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$

20 Integration

✓ Integrals of a function $y = f(x)$:

1. $\int f(x)\mathrm{d}x$: Indefinite integral of $f(x)$
2. $\int_a^b f(x)\mathrm{d}x$: Definite integral of $f(x)$ from a to b

✓ Rules of integration:

$\int x^n dx = \frac{1}{n+1}x^{n+1} + C$	$\int (p'(x)+q'(x))dx = p(x)+q(x)+C$
$\int \cos x dx = \sin x + C$	$\int cp'(x)dx = cp(x)+C$
$\int \sin x dx = -\cos x + C$	$\int_a^b f'(x)dx = [f(x)]_a^b = f(b)-f(a)$
$\int \frac{1}{\cos^2 x} dx = \tan x + C$	Integration by substitution
$\int e^x dx = e^x + C$	$\int \frac{1}{x} dx = \ln x + C$

✓ More integration rules:

$\int \sec^2 x dx = \tan x + C$	$\int \text{cosec}^2 x dx = -\cot x + C$
$\int \sec x \tan x dx = \sec x + C$	$\int \text{cosec} x \cot x dx = -\text{cosec} x + C$
$\int a^x dx = \frac{a^x}{\ln a} + C$	$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$
$\int \frac{1}{1+x^2} dx = \arctan x + C$	

✓ Integration by parts:

1. $\int u dv = uv - \int v du$
2. $\int_a^b u dv = [uv]_a^b - \int_a^b v du$

21 Applications of Integration

✓ Areas on x - y plane, between $x = a$ and $x = b$:

1. $\int_a^b f(x)dx$: Area under the graph of $f(x)$ and above the x-axis
2. $-\int_a^b f(x)dx$: Area under the x-axis and above the graph of $f(x)$
3. $\int_a^b (f(x)-g(x))dx$: Area under the graph of $f(x)$ and above the graph of $g(x)$

✓ Applications in kinematics:

1. $a(t)$: Acceleration with respect to the time t
2. $v(t)=\int a(t)\mathrm{d}t$: Velocity
3. $s(t)=\int v(t)\mathrm{d}t$: Displacement
4. $d=\int_{t_1}^{t_2}\left|v(t)\right|\mathrm{d}t$: Total distance travelled between t_1 and t_2

✓ Areas on x-y plane, between $y=c$ and $y=d$:

1. $\int_c^d g(y)\mathrm{d}y$: Area on the left of the graph of $g(y)$ and on the right of the y-axis
2. $-\int_c^d g(y)\mathrm{d}y$: Area on the left of the y-axis and on the right of the graph of $g(y)$
3. $\int_c^d (g(y)-f(y))\mathrm{d}y$: Area on the left of the graph of $g(y)$ and on the right of the graph of $f(y)$

✓ Volumes of revolutions about the x-axis, between $x=a$ and $x=b$:

1. $V=\pi\int_a^b (f(x))^2\mathrm{d}x$: Volume of revolution when the region between the graph of $f(x)$ and the x-axis is rotated 360° about the x-axis
2. $V=\pi\int_a^b ((f(x))^2-(g(x))^2)\mathrm{d}x$: Volume of revolution when the region between the graphs of $f(x)$ and $g(x)$ is rotated 360° about the x-axis

✓ Volumes of revolutions about the y-axis, between $y=c$ and $y=d$:

1. $V=\pi\int_c^d (g(y))^2\mathrm{d}y$: Volume of revolution when the region between the graph of $g(y)$ and the y-axis is rotated 360° about the y-axis
2. $V=\pi\int_c^d ((g(y))^2-(f(y))^2)\mathrm{d}y$: Volume of revolution when the region between the graphs of $g(y)$ and $f(y)$ is rotated 360° about the y-axis

22 Limits and Maclaurin Series

✓ L'Hôpital's Rule under the conditions of indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

✓ $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f^{(3)}(0)+\cdots$: Maclaurin series

✓ Common Maclaurin series:

1. $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$
2. $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$
3. $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$
4. $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$ for $-1<x\le 1$
5. $\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$ for $-1<x<1$
6. $(1+x)^n=1+nx+\frac{(n)(n-1)}{(2)(1)}x^2+\frac{(n)(n-1)(n-2)}{(3)(2)(1)}x^3+\cdots$ for $-1<x<1$

23 Differential Equations

✓ $\frac{\mathrm{d}y}{\mathrm{d}x}=f(x,y)$: First order differential equation

✓ Solving $\frac{\mathrm{d}y}{\mathrm{d}x}=f(x)g(y)$ by separating variables:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f(x)g(y)\Rightarrow\int\frac{1}{g(y)}\mathrm{d}y=\int f(x)\mathrm{d}x$$

✓ Solving $\frac{\mathrm{d}y}{\mathrm{d}x}+f(x)\cdot y=g(x,y)$ by integrating factor:

$e^{\int f(x)\mathrm{d}x}$: Integrating factor

$$\frac{\mathrm{d}y}{\mathrm{d}x}+f(x)\cdot y=g(x,y)\Rightarrow e^{\int f(x)\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x}+e^{\int f(x)\mathrm{d}x}\cdot f(x)\cdot y=e^{\int f(x)\mathrm{d}x}\cdot g(x,y)$$

✓ Solving $\frac{\mathrm{d}y}{\mathrm{d}x}=f(x,y)$ by Euler's method, with (x_0,y_0) and step length h:

$$\begin{cases} x_{n+1}=x_n+h \\ y_{n+1}=y_n+h\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x_n,\,y_n)} \end{cases}$$

✓ Developing a Maclaurin series from $\frac{\mathrm{d}y}{\mathrm{d}x}=f(x,y)$:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f(x,y)\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}f(x,y)\Rightarrow\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x,y)\right)$$

24 Statistics

✓ Relationship between frequencies and cumulative frequencies:

Data	Frequency	Data less than or equal to	Cumulative frequency
10	f_1	10	f_1
20	f_2	20	f_1+f_2
30	f_3	30	$f_1+f_2+f_3$

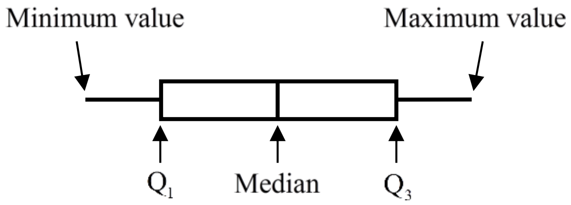
✓ Measures of central tendency for a data set $\{x_1, x_2, x_3, \cdots, x_n\}$ arranged in ascending order:

1. $\bar{x}=\frac{x_1+x_2+x_3+\cdots+x_n}{n}$: Mean
2. The datum or the average value of two data at the middle: Median
3. The datum appears the most: Mode

✓ Measures of dispersion for a data set $\{x_1, x_2, x_3, \cdots, x_n\}$ arranged in ascending order:

1. $x_n - x_1$: Range
2. Two subgroups A and B can be formed from the data set such that all data of the subgroup A are less than or equal to the median, while all data of the subgroup B are greater than or equal to the median
3. $Q_1 =$ The median of the subgroup A: Lower quartile
4. $Q_3 =$ The median of the subgroup B: Upper quartile
5. $Q_3 - Q_1$: Inter-quartile range (IQR)
6. $\sigma = \sqrt{\dfrac{(x_1-\overline{x})^2+(x_2-\overline{x})^2+(x_3-\overline{x})^2+\cdots+(x_n-\overline{x})^2}{n}}$: Standard deviation

✓ Box-and-whisker diagram:



✓ A datum x is defined to be an outlier if $x < Q_1 - 1.5\text{IQR}$ or $x > Q_3 + 1.5\text{IQR}$

✓ Coding of data:

1. Only the mean, the median, the mode and the quartiles will change when each datum of the data set is added or subtracted by a value
2. All measures of central tendency and measures of dispersion will change when each datum of the data set is multiplied or divided by a value

25 Permutations and Combinations

✓ Permutations and combinations when a sample of r objects are selected from a set of n objects, $0 \le r \le n$:

1. ${}^nP_r = \dfrac{n!}{(n-r)!}$: Number of permutations when the order is taken into account
2. ${}^nC_r = \dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$: Number of combinations when the order is not taken into account

26 Probability

✓ Terminologies:

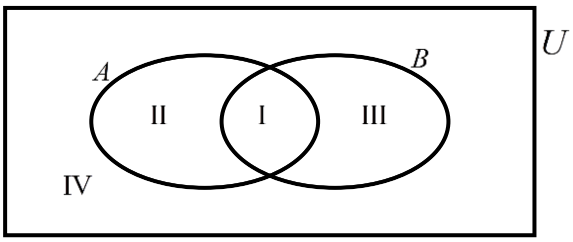
1. U: Universal set
2. A: Event
3. x: Outcome of an event
4. $n(U)$: Total number of elements
5. $n(A)$: Number of elements in A

✓ Formulae for probability:

1. $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$
2. $\mathrm{P}(A') = 1 - \mathrm{P}(A)$
3. $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$
4. $\mathrm{P}(A) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B')$
5. $\mathrm{P}(A' \cap B') + \mathrm{P}(A \cup B) = 1$
6. $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$ and $\mathrm{P}(A \cap B) = 0$ if A and B are mutually exclusive
7. $\mathrm{P}(A \cap B) = \mathrm{P}(A) \cdot \mathrm{P}(B)$ and $\mathrm{P}(A \mid B) = \mathrm{P}(A)$ if A and B are independent

✓ Venn diagram:

1. Region I: $A \cap B$
2. Region II: $A \cap B'$
3. Region III: $A' \cap B$
4. Region IV: $(A \cup B)'$



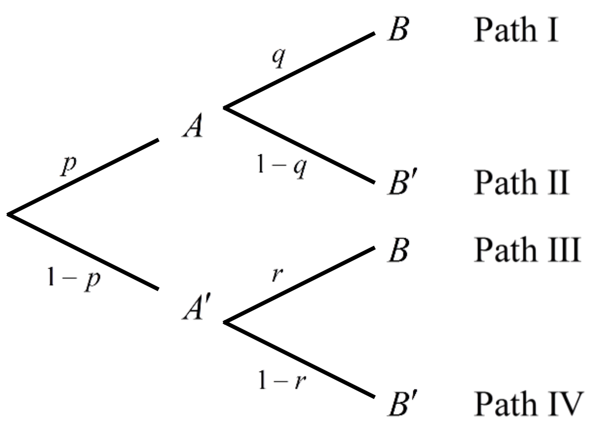
✓ Tree diagram:

1. Path I: $P(A \cap B) = pq$
2. Path I + Path III:

$= P(B)$

$= P(A \cap B) + P(A' \cap B)$

$= pq + (1 - p)r$



✓ Bayes' theorem:

1. $P(A|B) = \dfrac{P(A)P(B|A)}{P(A)P(B|A) + P(A')P(B|A')}$ for two events
2. $P(A_i|B) = \dfrac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)}$ $(i = 1, 2, 3)$ for three events

27 Discrete Probability Distributions

✓ Properties of a discrete random variable X:

X	x_1	x_2	...	x_n
$P(X = x)$	$P(X = x_1)$	$P(X = x_2)$	...	$P(X = x_n)$

1. $P(X = x_1) + P(X = x_2) + \cdots + P(X = x_n) = 1$
2. $E(X) = x_1 P(X = x_1) + x_2 P(X = x_2) + \cdots + x_n P(X = x_n)$: Expected value of X
3. $E(X) = 0$ if a fair game is considered

✓ Properties of a discrete random variable X:

X	x_1	x_2	...	x_n
$P(X = x)$	$P(X = x_1)$	$P(X = x_2)$	...	$P(X = x_n)$

1. $E(X^2) = x_1^2 P(X = x_1) + x_2^2 P(X = x_2) + \cdots + x_n^2 P(X = x_n)$
2. $\text{Var}(X) = E(X^2) - (E(X))^2$: Variance of X

✓ Linear transformation of a random variable X:

1. $E(aX + b) = aE(X) + b$: Expected value of X
2. $\text{Var}(aX + b) = a^2 \text{Var}(X)$

28 Binomial Distribution

✓ Properties of a random variable $X \sim \mathrm{B}(n, p)$ following binomial distribution:

1. Only two outcomes from every independent trial (Success and failure)
2. n : Number of trials
3. p : Probability of success
4. X : Number of successes in n trials

✓ Formulae for binomial distribution:

1. $\mathrm{P}(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$ for $0 \le r \le n$, $r \in \mathbb{Z}$
2. $\mathrm{E}(X) = np$: Expected value of X
3. $\mathrm{Var}(X) = np(1-p)$: Variance of X
4. $\sqrt{np(1-p)}$: Standard deviation of X
5. $\mathrm{P}(X \le r) = \mathrm{P}(X < r+1) = 1 - \mathrm{P}(X \ge r+1)$

29 Continuous Probability Distributions

✓ Properties of a continuous random variable X :

$$p(x) = \begin{cases} f(x) & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

1. $\int_a^b f(x)\mathrm{d}x = 1$
2. $\mathrm{E}(X) = \int_a^b x \cdot f(x)\mathrm{d}x$: Expected value of X
3. Q_2 : Median of X , which is the solution of the equation $\int_a^{Q_2} f(x)\mathrm{d}x = 0.5$
4. Q_1 : Lower quartile of X , which is the solution of $\int_a^{Q_1} f(x)\mathrm{d}x = 0.25$
5. Q_3 : Upper quartile of X , which is the solution of $\int_a^{Q_3} f(x)\mathrm{d}x = 0.75$
6. The maximum value of $f(x)$ is the mode of X
7. $\mathrm{E}(X^2) = \int_a^b x^2 \cdot f(x)\mathrm{d}x$

30 Normal Distribution

✓ Properties of a random variable $X \sim \mathrm{N}(\mu, \sigma^2)$ following normal distribution:

1. μ : Mean
2. σ : Standard deviation
3. The mean, the median and the mode are the same
4. The normal curve representing the distribution is a bell-shaped curve which is symmetric about the middle vertical line
5. $\mathrm{P}(X < \mu) = \mathrm{P}(X > \mu) = 0.5$
6. The total area under the curve is 1

✓ Standardization of a normal variable:

1. $Z \sim \mathrm{N}(0, 1^2)$: Standard normal distribution with mean 0 and standard deviation 1
2. $Z = \dfrac{X - \mu}{\sigma}$ for $X \sim \mathrm{N}(\mu, \sigma^2)$

31 Bivariate Analysis

✓ Correlations:

Positive	Strong	$0.75 < r < 1$
	Moderate	$0.5 < r < 0.75$
	Weak	$0 < r < 0.5$
No		$r = 0$
Negative	Weak	$-0.5 < r < 0$
	Moderate	$-0.75 < r < -0.5$
	Strong	$-1 < r < -0.75$

where r is the correlation coefficient

✓ Linear regression:

1. $y = ax + b$: Regression line of y on x
2. $x = ay + b$: Regression line of x on y

32 Paper 3 Analysis

✓ Nature of paper: Structured question

✓ Time allowed: 60 minutes

✓ Maximum mark: 55 marks

✓ Number of questions: 2

✓ Mark range per question: 25 marks to 30 marks

✓ Weighting: 20% of the total mark

✓ Ways of assessing:

1. Find
   (a) the value of a quantity
   (b) the formula of a quantity
   (c) an inequality connecting quantities
   (d) the limit of a quantity
2. Show
   (a) a quantity equals to a value
   (b) the formula of a quantity
   (c) the limit of a quantity
   (d) the recurrence relation of a quantity
3. Solve an equation
4. Geometrically interpret a result
5. Sketch a graph
6. Plot and label a quantity on a diagram
7. Suggest an expression for a quantity
8. Express the formula of a quantity
9. Verify
   (a) the value of a quantity
   (b) the trueness of a statement
10. Prove the trueness of a statement
11. Explain the trueness of a statement

Notes