

Solución de Práctica de Prueba 3 de AE NS Set 4

1. (a) $F(2)$

$$= \sum_{r=1}^2 \text{sen} \frac{\pi}{2(2)} \text{sen} \frac{r\pi}{2} \quad \text{(M1) por sustitución}$$

$$= \text{sen} \frac{\pi}{4} \sum_{r=1}^2 \text{sen} \frac{r\pi}{2}$$

$$= \text{sen} \frac{\pi}{4} \left(\text{sen} \frac{\pi}{2} + \text{sen} \pi \right) \quad \text{A1}$$

$$= \left(\text{sen} \frac{\pi}{4} \right) (1 + 0)$$

$$= \text{sen} \frac{\pi}{4} \quad \text{A1}$$

[3]

(b) (i) $\cos(x-y) - \cos(x+y)$
 $= \cos x \cos y + \text{sen} x \text{sen} y$
 $-(\cos x \cos y - \text{sen} x \text{sen} y)$
 $= 2 \text{sen} x \text{sen} y$

A2

AG

(ii) Sean $x = \frac{A+B}{2}$ y $y = \frac{B-A}{2}$.

$$\cos(x-y) - \cos(x+y)$$

$$= \cos \left(\frac{A+B}{2} - \frac{B-A}{2} \right)$$

$$- \cos \left(\frac{A+B}{2} + \frac{B-A}{2} \right)$$

$$= \cos \frac{2A}{2} - \cos \frac{2B}{2} \quad \text{M1}$$

$$= \cos A - \cos B$$

$$\therefore \cos A - \cos B = 2 \text{sen} \frac{A+B}{2} \text{sen} \frac{B-A}{2} \quad \text{AG}$$

[4]

(c) $F(4)$

$$= \sum_{r=1}^4 \text{sen} \frac{\pi}{2(4)} \text{sen} \frac{r\pi}{4}$$

$$= \text{sen} \frac{\pi}{8} \sum_{r=1}^4 \text{sen} \frac{r\pi}{4}$$

$$= \text{sen} \frac{\pi}{8} \left(\text{sen} \frac{\pi}{4} + \text{sen} \frac{\pi}{2} + \text{sen} \frac{3\pi}{4} + \text{sen} \pi \right) \quad \text{A1}$$

$$= \text{sen} \frac{\pi}{8} \text{sen} \frac{\pi}{4} + \text{sen} \frac{\pi}{8} \text{sen} \frac{\pi}{2}$$

$$+ \text{sen} \frac{\pi}{8} \text{sen} \frac{3\pi}{4} + \text{sen} \frac{\pi}{8} \text{sen} \pi$$

$$= \frac{1}{2} \left(\cos \left(\frac{\pi}{8} - \frac{\pi}{4} \right) - \cos \left(\frac{\pi}{8} + \frac{\pi}{4} \right) \right)$$

$$+ \frac{1}{2} \left(\cos \left(\frac{\pi}{8} - \frac{\pi}{2} \right) - \cos \left(\frac{\pi}{8} + \frac{\pi}{2} \right) \right) \quad \text{M1A1}$$

$$+ \frac{1}{2} \left(\cos \left(\frac{\pi}{8} - \frac{3\pi}{4} \right) - \cos \left(\frac{\pi}{8} + \frac{3\pi}{4} \right) \right)$$

$$= \frac{1}{2} \left(\cos \left(-\frac{\pi}{8} \right) - \cos \frac{3\pi}{8} + \cos \left(-\frac{3\pi}{8} \right) - \cos \frac{5\pi}{8} \right. \\ \left. + \cos \left(-\frac{5\pi}{8} \right) - \cos \frac{7\pi}{8} \right)$$

$$= \frac{1}{2} \left(\cos \frac{\pi}{8} - \cos \frac{3\pi}{8} + \cos \frac{3\pi}{8} - \cos \frac{5\pi}{8} \right. \\ \left. + \cos \frac{5\pi}{8} - \cos \frac{7\pi}{8} \right) \quad \text{A1}$$

$$= \frac{1}{2} \left(\cos \frac{\pi}{8} - \cos \frac{7\pi}{8} \right)$$

$$= \frac{1}{2} \left(2 \text{sen} \frac{\frac{\pi}{8} + \frac{7\pi}{8}}{2} \text{sen} \frac{\frac{7\pi}{8} - \frac{\pi}{8}}{2} \right) \quad \text{A1}$$

$$= \text{sen} \frac{\pi}{2} \text{sen} \frac{3\pi}{8}$$

$$= \text{sen} \frac{3\pi}{8} \quad \text{A1}$$

[6]

(d) $F(n)$

$$= \sum_{r=1}^n \operatorname{sen} \frac{\pi}{2n} \operatorname{sen} \frac{r\pi}{n}$$
$$= \sum_{r=1}^n \frac{1}{2} \left(\cos \left(\frac{\pi}{2n} - \frac{r\pi}{n} \right) - \cos \left(\frac{\pi}{2n} + \frac{r\pi}{n} \right) \right) \quad \text{M1A1}$$

$$= \sum_{r=1}^n \frac{1}{2} \left(\cos \left(\frac{\pi}{2n} - \frac{2r\pi}{2n} \right) - \cos \left(\frac{\pi}{2n} + \frac{2r\pi}{2n} \right) \right)$$
$$= \sum_{r=1}^n \frac{1}{2} \left(\cos \frac{(1-2r)\pi}{2n} - \cos \frac{(1+2r)\pi}{2n} \right) \quad \text{M1}$$

$$= \frac{1}{2} \left(\cos \frac{(1-2(1))\pi}{2n} - \cos \frac{(1+2(1))\pi}{2n} \right. \\ \left. + \cos \frac{(1-2(2))\pi}{2n} - \cos \frac{(1+2(2))\pi}{2n} \right. \\ \left. + \dots + \cos \frac{(1-2n)\pi}{2n} - \cos \frac{(1+2n)\pi}{2n} \right)$$

$$= \frac{1}{2} \left(\cos \left(-\frac{\pi}{2n} \right) - \cos \frac{3\pi}{2n} + \cos \left(-\frac{3\pi}{2n} \right) - \cos \frac{5\pi}{2n} \right. \\ \left. + \dots + \cos \frac{(1-2n)\pi}{2n} - \cos \frac{(1+2n)\pi}{2n} \right)$$

$$= \frac{1}{2} \left(\cos \frac{\pi}{2n} - \cos \frac{3\pi}{2n} + \cos \frac{3\pi}{2n} - \cos \frac{5\pi}{2n} \right. \\ \left. + \dots + \cos \frac{(2n-1)\pi}{2n} - \cos \frac{(1+2n)\pi}{2n} \right) \quad \text{A1}$$

$$= \frac{1}{2} \left(\cos \frac{\pi}{2n} - \cos \frac{(1+2n)\pi}{2n} \right) \quad \text{M1}$$

$$= \frac{1}{2} \left(2 \operatorname{sen} \frac{\frac{\pi}{2n} + \frac{(1+2n)\pi}{2n}}{2} \operatorname{sen} \frac{\frac{(1+2n)\pi}{2n} - \frac{\pi}{2n}}{2} \right) \quad \text{A1}$$

$$= \operatorname{sen} \frac{(2+2n)\pi}{4n} \operatorname{sen} \frac{2n\pi}{4n}$$

$$= \operatorname{sen} \frac{(1+n)\pi}{2n} \operatorname{sen} \frac{\pi}{2}$$

$$\therefore F(n) = \operatorname{sen} \frac{(1+n)\pi}{2n} \quad \text{AG}$$

[6]

$$\begin{aligned}
\text{(e)} \quad & |z^r - 1| \\
& = \left| \left(\cos \frac{2\pi}{n} + i \operatorname{sen} \frac{2\pi}{n} \right)^r - 1 \right| \\
& = \left| \cos \frac{2\pi r}{n} + i \operatorname{sen} \frac{2\pi r}{n} - 1 \right| && \text{(M1) por enfoque válido} \\
& = \sqrt{\left(\cos \frac{2\pi r}{n} - 1 \right)^2 + \operatorname{sen}^2 \frac{2\pi r}{n}} \\
& = \sqrt{\cos^2 \frac{2\pi r}{n} - 2 \cos \frac{2\pi r}{n} + 1 + \operatorname{sen}^2 \frac{2\pi r}{n}} && \text{M1} \\
& = \sqrt{2 - 2 \cos \frac{2\pi r}{n}} \\
& = \sqrt{2 - 2 \left(1 - 2 \operatorname{sen}^2 \frac{\pi r}{n} \right)} && \text{A1} \\
& = \sqrt{4 \operatorname{sen}^2 \frac{\pi r}{n}} \\
& = 2 \operatorname{sen} \frac{\pi r}{n} \\
& \because -1 \leq \operatorname{sen} \frac{\pi r}{n} \leq 1 && \text{R1} \\
& \therefore |z^r - 1| \leq 2 && \text{A1}
\end{aligned}$$

[5]

$$\begin{aligned}
\text{(f)} \quad G(n) &= \sum_{r=1}^n |z^r - 1| \\
&= \sum_{r=1}^n 2 \operatorname{sen} \frac{\pi r}{n} \\
&= \frac{2 \sum_{r=1}^n \operatorname{sen} \frac{\pi}{2n} \operatorname{sen} \frac{\pi r}{n}}{\operatorname{sen} \frac{\pi}{2n}} && \text{M1} \\
&= \frac{2F(n)}{\operatorname{sen} \frac{\pi}{2n}} && \text{A1} \\
&= \frac{2 \operatorname{sen} \frac{(1+n)\pi}{2n}}{\operatorname{sen} \frac{\pi}{2n}} \\
&= \frac{2 \cos \left(\frac{\pi}{2} - \frac{(1+n)\pi}{2n} \right)}{\operatorname{sen} \frac{\pi}{2n}} && \text{A1} \\
&= \frac{2 \cos \left(\frac{n\pi}{2n} - \frac{\pi + n\pi}{2n} \right)}{\operatorname{sen} \frac{\pi}{2n}} \\
&= \frac{2 \cos \left(-\frac{\pi}{2n} \right)}{\operatorname{sen} \frac{\pi}{2n}} && \text{M1} \\
&= \frac{2 \cos \frac{\pi}{2n}}{\operatorname{sen} \frac{\pi}{2n}} \\
&= 2 \cot \frac{\pi}{2n} && \text{A1}
\end{aligned}$$

[5]

2. (a) (i) $I(0)$

$$= \int_0^{\pi} x dx \quad \text{M1}$$

$$= \left[\frac{1}{2} x^2 \right]_0^{\pi} \quad \text{A1}$$

$$= \frac{1}{2} \pi^2 - \frac{1}{2} (0)^2$$

$$= \frac{1}{2} \pi^2 \quad \text{AG}$$

(ii) $I(1)$

$$= \int_0^{\pi} x \operatorname{sen} x dx$$

Sea $\theta = \cos x$. (M1) por enfoque válido

$$\frac{d\theta}{dx} = -\operatorname{sen} x \Rightarrow (-1) \frac{d\theta}{dx} = \operatorname{sen} x$$

$\therefore I(1)$

$$= \int_0^{\pi} x(-1) \frac{d(\cos x)}{dx} dx$$

$$= [-x \cos x]_0^{\pi} - \int_0^{\pi} \cos x \cdot \frac{d(-x)}{dx} dx \quad \text{A1}$$

$$= [-x \cos x]_0^{\pi} - \int_0^{\pi} \cos x \cdot (-1) dx \quad \text{A1}$$

$$= [-x \cos x]_0^{\pi} + \int_0^{\pi} \cos x dx$$

$$= [-x \cos x]_0^{\pi} + [\operatorname{sen} x]_0^{\pi} \quad \text{A1}$$

$$= [-x \cos x + \operatorname{sen} x]_0^{\pi}$$

$$= (-\pi \cos \pi + \operatorname{sen} \pi) - (0 + \operatorname{sen} 0)$$

$$= \pi \quad \text{A1}$$

(b) (i) $I(n+2)$

$$= \int_0^{\pi} x \operatorname{sen}^{n+2} x dx$$

$$= \int_0^{\pi} x \operatorname{sen}^n x \operatorname{sen}^2 x dx \quad \text{M1}$$

$$= \int_0^{\pi} x \operatorname{sen}^n x (1 - \cos^2 x) dx$$

$$= \int_0^{\pi} x \operatorname{sen}^n x dx - \int_0^{\pi} x \operatorname{sen}^n x \cos^2 x dx \quad \text{A1}$$

$$= I(n) - \int_0^{\pi} x \operatorname{sen}^n x \cos^2 x dx \quad \text{AG}$$

[7]

$$\begin{aligned}
 \text{(ii)} \quad & \int_0^\pi x \operatorname{sen}^n x \cos^2 x dx \\
 &= \frac{1}{n+1} \int_0^\pi x \cos x \cdot \frac{d(\operatorname{sen}^{n+1} x)}{dx} dx \\
 &= \frac{1}{n+1} \left\{ \left[x \cos x \operatorname{sen}^{n+1} x \right]_0^\pi - \int_0^\pi \operatorname{sen}^{n+1} x \cdot \frac{d(x \cos x)}{dx} dx \right\} \quad \text{A1}
 \end{aligned}$$

$$= \frac{1}{n+1} \left\{ \left[x \cos x \operatorname{sen}^{n+1} x \right]_0^\pi - \int_0^\pi \operatorname{sen}^{n+1} x (\cos x - x \operatorname{sen} x) dx \right\} \quad \text{A1}$$

$$= \frac{1}{n+1} \left\{ \left[x \cos x \operatorname{sen}^{n+1} x \right]_0^\pi - \int_0^\pi (\operatorname{sen}^{n+1} x \cos x - x \operatorname{sen}^{n+2} x) dx \right\}$$

$$= \frac{1}{n+1} \left\{ \left[x \cos x \operatorname{sen}^{n+1} x \right]_0^\pi - \int_0^\pi \operatorname{sen}^{n+1} x \cos x dx + I(n+2) \right\} \quad \text{M1}$$

$$= \frac{1}{n+1} \left\{ (\pi \cos \pi \operatorname{sen}^{n+1} \pi - 0) - \int_0^\pi \operatorname{sen}^{n+1} x \cos x dx + I(n+2) \right\} \quad \text{M1}$$

$$= \frac{1}{n+1} \left\{ - \int_0^\pi \operatorname{sen}^{n+1} x \cos x dx + I(n+2) \right\}$$

Sea $u = \operatorname{sen} x$. (M1) por sustitución

$$\frac{du}{dx} = \cos x \Rightarrow du = \cos x dx$$

$$x = \pi \Rightarrow u = \operatorname{sen} \pi = 0$$

$$x = 0 \Rightarrow u = \operatorname{sen} 0 = 0$$

$$\therefore \int_0^\pi x \operatorname{sen}^n x \cos^2 x dx$$

$$= \frac{1}{n+1} \left\{ - \int_0^0 u^{n+1} du + I(n+2) \right\} \quad \text{A1}$$

$$= \frac{1}{n+1} (0 + I(n+2))$$

$$= \frac{1}{n+1} I(n+2) \quad \text{A1}$$

$$(iii) \quad I(n+2) = I(n) - \frac{1}{n+1} I(n+2) \quad A1$$

$$(n+1)I(n+2) = (n+1)I(n) - I(n+2)$$

$$(n+2)I(n+2) = (n+1)I(n) \quad M1$$

$$I(n+2) = \frac{n+1}{n+2} I(n) \quad AG$$

[11]

(c) (i) $I(4)$

$$= \frac{2+1}{2+2} I(2) \quad M1$$

$$= \frac{3}{4} \left(\frac{0+1}{0+2} I(0) \right) \quad M1$$

$$= \frac{3}{4} \left(\frac{1}{2} \cdot \frac{1}{2} \pi^2 \right)$$

$$= \frac{3}{16} \pi^2 \quad A1$$

(ii) $I(7)$

$$= \frac{5+1}{5+2} I(5) \quad M1$$

$$= \frac{6}{7} \left(\frac{3+1}{3+2} I(3) \right) \quad M1$$

$$= \frac{6}{7} \left(\frac{4}{5} \right) \left(\frac{1+1}{1+2} I(1) \right)$$

$$= \frac{6}{7} \left(\frac{4}{5} \right) \left(\frac{2}{3} \pi \right)$$

$$= \frac{16}{35} \pi \quad A1$$

(d) $0 \leq \text{sen } x \leq 1$ for $0 \leq x \leq \pi$. A1

Por lo tanto, $\text{sen}^2 x \leq \text{sen } x \leq 1$, implica que

$$\int_0^\pi x \text{sen}^{2n-2} x \cdot \text{sen}^2 x dx$$

$$\leq \int_0^\pi x \text{sen}^{2n-2} x \cdot \text{sen } x dx \leq \int_0^\pi x \text{sen}^{2n-2} x \cdot 1 dx \quad R1$$

Por lo tanto, $I(2n) \leq I(2n-1) \leq I(2n-2)$ para $n \geq 1$. AG

[6]

[2]