Chapter 11 Solution

Exercise 45

1. $y\ln x = x^2 - y$

$\frac{d}{dx}(y\ln x) = \frac{d}{dx}(x^2 - y)$ M1

$\left(\frac{dy}{dx}\right)(\ln x) + (y)\left(\frac{1}{x}\right) = 2x - \frac{dy}{dx}$ A2

$\ln x\frac{dy}{dx} + \frac{y}{x} = 2x - \frac{dy}{dx}$

$\ln x\frac{dy}{dx} + \frac{dy}{dx} = 2x - \frac{y}{x}$

$(\ln x + 1)\frac{dy}{dx} = 2x - \frac{y}{x}$

$\frac{dy}{dx} = \frac{2x - \frac{y}{x}}{\ln x + 1}$ A1

$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{2(1) - \frac{1}{1}}{\ln 1 + 1}$ M1

$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{2-1}{0+1}$

$\left.\frac{dy}{dx}\right|_{(1,1)} = 1$ A1

The equation of tangent:

$y - 1 = 1(x - 1)$ A1

$y - 1 = x - 1$

$y = x$ AG

[7]

2. $2y\arctan x = \frac{5}{4}x + \frac{\pi - 5}{4}$

$\frac{d}{dx}(2y\arctan x) = \frac{d}{dx}\left(\frac{5}{4}x\right) + \frac{d}{dx}\left(\frac{\pi-5}{4}\right)$ (M1) for valid approach

$\left(2\frac{dy}{dx}\right)(\arctan x) + (2y)\left(\frac{1}{1+x^2}\right) = \frac{5}{4} - 0$ (A2) for correct approach

$2\arctan x\frac{dy}{dx} + \frac{2y}{1+x^2} = \frac{5}{4}$

$2\arctan x\frac{dy}{dx} = \frac{5}{4} - \frac{2y}{1+x^2}$

$\frac{dy}{dx} = \frac{1}{2\arctan x}\left(\frac{5}{4} - \frac{2y}{1+x^2}\right)$

$\left.\frac{dy}{dx}\right|_{\left(1,\frac{1}{2}\right)} = \frac{1}{2\arctan 1}\left(\frac{5}{4} - \frac{2\left(\frac{1}{2}\right)}{1+1^2}\right)$ (M1) for substitution

$\left.\frac{dy}{dx}\right|_{\left(1,\frac{1}{2}\right)} = \frac{1}{2\left(\frac{\pi}{4}\right)}\left(\frac{5}{4} - \frac{1}{2}\right)$

$\left.\frac{dy}{dx}\right|_{\left(1,\frac{1}{2}\right)} = \frac{2}{\pi}\left(\frac{3}{4}\right)$

$\left.\frac{dy}{dx}\right|_{\left(1,\frac{1}{2}\right)} = \frac{3}{2\pi}$ A1

Thus, the slope of normal

$= -1 \div \frac{3}{2\pi}$

$= -\frac{2\pi}{3}$ A1

The equation of normal:

$y - \frac{1}{2} = -\frac{2\pi}{3}(x-1)$ (A1) for substitution

$y - \frac{1}{2} = -\frac{2\pi}{3}x + \frac{2\pi}{3}$

$6y - 3 = -4\pi x + 4\pi$

$4\pi x + 6y - (4\pi + 3) = 0$ A1

[8]

3. $x^2 + 4y^2 = 100$

$\frac{d}{dx}(x^2) + \frac{d}{dx}(4y^2) = \frac{d}{dx}(100)$ (M1) for valid approach

$2x + 8y\frac{dy}{dx} = 0$ (A1) for correct approach

$8y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = -\frac{x}{4y}$ A1

$\therefore -\frac{x}{4y} = -\frac{3}{8}$ (M1) for setting equation

$x = \frac{3}{2}y$

$\therefore \left(\frac{3}{2}y\right)^2 + 4y^2 = 100$

$\frac{9}{4}y^2 + 4y^2 = 100$

$\frac{25}{4}y^2 = 100$

$y^2 = 16$

$y = -4$ or $y = 4$ A1

When $y = -4$, $x = \frac{3}{2}(-4) = -6$.

When $y = 4$, $x = \frac{3}{2}(4) = 6$.

The equations of tangent:

$y - (-4) = -\frac{3}{8}(x - (-6))$ or $y - 4 = -\frac{3}{8}(x - 6)$ (A1) for substitution

$8(y + 4) = -3(x + 6)$ or $8(y - 4) = -3(x - 6)$

$8y + 32 = -3x - 18$ or $8y - 32 = -3x + 18$

$3x + 8y + 50 = 0$ or $3x + 8y - 50 = 0$ A2

[8]

4. $9x^2 + y^2 = 106$

$\frac{d}{dx}(9x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(106)$ (M1) for valid approach

$18x + 2y\frac{dy}{dx} = 0$ (A1) for correct approach

$2y\frac{dy}{dx} = -18x$

$\frac{dy}{dx} = -\frac{9x}{y}$ A1

$\therefore -\frac{9x}{y} \times \frac{5}{27} = -1$ (M1) for setting equation

$-\frac{5x}{3y} = -1$

$x = \frac{3}{5}y$

$\therefore 9\left(\frac{3}{5}y\right)^2 + y^2 = 106$

$\frac{81}{25}y^2 + y^2 = 106$

$\frac{106}{25}y^2 = 106$

$y^2 = 25$

$y = -5$ or $y = 5$ A1

When $y = -5$, $x = \frac{3}{5}(-5) = -3$.

When $y = 5$, $x = \frac{3}{5}(5) = 3$.

The equations of normal:

$y - (-5) = \frac{5}{27}(x - (-3))$ or $y - 5 = \frac{5}{27}(x - 3)$ (A1) for substitution

$27(y + 5) = 5(x + 3)$ or $27(y - 5) = 5(x - 3)$

$27y + 135 = 5x + 15$ or $27y - 135 = 5x - 15$

$5x - 27y - 120 = 0$ or $5x - 27y + 120 = 0$ A2

[8]

Exercise 46

1. (a) $f(a)=\arctan a$

$\arctan a+\ln(1+a^2)=\arctan a$ (M1) for setting equation

$\ln(1+a^2)=0$

$1+a^2=1$

$a^2=0$

$a=0$ A1

[2]

(b) $f'(x)=\frac{1}{1+x^2}+\left(\frac{1}{1+x^2}\right)(2x)$ (A1) for correct approach

$f'(x)=\frac{1+2x}{1+x^2}$ A1

[2]

(c) $f'(x)=0$

$\therefore \frac{1+2x}{1+x^2}=0$ M1

$1+2x=0$

$2x=-1$

$x=-\frac{1}{2}$

By the first derivative test, M1A1

x	$x<-\frac{1}{2}$	$x=-\frac{1}{2}$	$x>-\frac{1}{2}$
$f'(x)$	–	0	+

Thus, there is no local maximum of $f(x)$ for $x\in\mathbb{R}$. AG

[3]

(d) By the first derivative test, $f(x)$ attains its minimum at $x=-\frac{1}{2}$. (R1) for correct argument

$$f\left(-\frac{1}{2}\right)=\arctan\left(-\frac{1}{2}\right)+\ln\left(1+\left(-\frac{1}{2}\right)^2\right)$$

$$f\left(-\frac{1}{2}\right)=\arctan\left(-\frac{1}{2}\right)+\ln\frac{5}{4}$$

Thus, the coordinates of the local minimum of $f(x)$ are $\left(-\frac{1}{2},\arctan\left(-\frac{1}{2}\right)+\ln\frac{5}{4}\right)$. A1

[2]

(e) $$f''(x)=\frac{(1+x^2)(2)-(1+2x)(2x)}{(1+x^2)^2}$$ (A1) for correct approach

$$f''(x)=\frac{2+2x^2-2x-4x^2}{(1+x^2)^2}$$

$$f''(x)=\frac{2-2x-2x^2}{(1+x^2)^2}$$

$$f''(x)=\frac{2(1-x-x^2)}{(1+x^2)^2}$$ A1

[2]

(f) $f''(x)=0$

$\therefore \frac{2(1-x-x^2)}{(1+x^2)^2}=0$ M1

$2(1-x-x^2)=0$

$x^2+x-1=0$ A1

$x=\frac{-1\pm\sqrt{1^2-4(1)(-1)}}{2(1)}$

$x=\frac{-1+\sqrt{5}}{2}$ or $x=\frac{-1-\sqrt{5}}{2}$ A1

x	$x<\frac{-1-\sqrt{5}}{2}$	$x=\frac{-1-\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}<x<\frac{-1+\sqrt{5}}{2}$	$x=\frac{-1+\sqrt{5}}{2}$	$x>\frac{-1+\sqrt{5}}{2}$
$f''(x)$	–	0	+	0	–

$f''(x)$ changes its sign at $x=\frac{-1-\sqrt{5}}{2}$ and

$x=\frac{-1+\sqrt{5}}{2}$. M1

Thus, there are two points of inflexion of $f(x)$ for

$x\in\mathbb{R}$. AG

[4]

(g) $f(-x)=0$ has the solutions $x=0$ and $x=1.17$. (A1) for correct values

The x-coordinate of the local minimum of $f(-x)$

is $\frac{1}{2}$.

$\therefore x\le 0$ or $x\ge 1.17$. A2

[3]

2. (a) $y=2$ A1

[1]

(b) $f'(x)=0-e^{-\frac{1}{2}x^2}(-x)$ (A1) for correct approach

$f'(x)=xe^{-\frac{1}{2}x^2}$ A1

[2]

(c) $f'(x)=0$

$\therefore xe^{-\frac{1}{2}x^2}=0$ M1

$x=0$

By the first derivative test, M1A1

x	$x<0$	$x=0$	$x>0$
$f'(x)$	–	0	+

Thus, there is no local maximum of $f(x)$ for $x\in\mathbb{R}$. AG

[3]

(d) By the first derivative test, $f(x)$ attains its minimum at $x=0$. (R1) for correct argument

$f(0)=2-e^{-\frac{1}{2}(0)^2}$ (M1) for substitution

$f(0)=2-1$

$f(0)=1$

Thus, the range of $f(x)$ is $1\le y<2$. A1

[3]

(e) $f''(x)=(1)\left(e^{-\frac{1}{2}x^2}\right)+(x)\left(e^{-\frac{1}{2}x^2}\right)(-x)$ (A1) for correct approach

$f''(x)=(1-x^2)e^{-\frac{1}{2}x^2}$ A1

[2]

(f) $f''(x)=0$

$\therefore (1-x^2)e^{-\frac{1}{2}x^2}=0$ M1

$1-x^2=0$

$x^2=1$

$x=-1$ or $x=1$ A1

x	$x<-1$	$x=-1$	$-1<x<1$	$x=1$	$x>1$
$f''(x)$	–	0	+	0	–

$f''(x)$ changes its sign at $x=-1$ and $x=1$. M1

Thus, there are two points of inflexion of $f(x)$ for $x\in\mathbb{R}$. AG

[3]

(g) The y-coordinate of the points of inflexion of $f(x)$

$= 2 - e^{-\frac{1}{2}(1)^2}$ (M1) for substitution

$= 2 - e^{-\frac{1}{2}}$

The y-coordinate of the points of inflexion of

$g(x) = \sqrt{e} f(x) + k$

$= \sqrt{e}\left(2 - e^{-\frac{1}{2}}\right) + k$ (A1) for correct approach

$= 2\sqrt{e} - 1 + k$

$\therefore 2\sqrt{e} - 1 + k = 0$

$k = 1 - 2\sqrt{e}$ A1

[3]

3. (a) $x=-6,\ x=6$ A2

[2]

(b) $f'(x)=\dfrac{(x^2-36)(1)-(x)(2x)}{(x^2-36)^2}$ (A1) for correct approach

$f'(x)=\dfrac{x^2-36-2x^2}{(x^2-36)^2}$

$f'(x)=-\dfrac{36+x^2}{(x^2-36)^2}$ A1

[2]

(c) $f'(x)=0$

$\therefore -\dfrac{36+x^2}{(x^2-36)^2}=0$ M1

$36+x^2=0$

$x^2=-36$

$\therefore f'(x)\neq 0$ for $x\neq -6,\ x\neq 6$. A1

Thus, there is no local extrema of $f(x)$. AG

[2]

(d) $f''(x)=-\dfrac{(x^2-36)^2(2x)-(36+x^2)(2)(x^2-36)(2x)}{((x^2-36)^2)^2}$ (A1) for correct approach

$f''(x)=-\dfrac{2x(x^2-36)^2-4x(36+x^2)(x^2-36)}{((x^2-36)^2)^2}$

$f''(x)=\dfrac{-2x(x^2-36)+4x(36+x^2)}{(x^2-36)^3}$ (M1) for simplification

$f''(x)=\dfrac{-2x^3+72x+144x+4x^3}{(x^2-36)^3}$

$f''(x)=\dfrac{2x^3+216x}{(x^2-36)^3}$

$f''(x)=\dfrac{2x(x^2+108)}{(x^2-36)^3}$ A1

[3]

(e) $f''(x)=0$

$\therefore \frac{2x(x^2+108)}{(x^2-36)^3}=0$ M1

$2x(x^2+108)=0$

$x=0$ A1

x	$x<0$	$x=0$	$x>0$
$f''(x)$	–	0	+

$f''(x)$ changes its sign at $x=0$ only. M1

Thus, there is only one point of inflexion of $f(x)$. AG

[3]

(f) Let $\frac{x}{x^2-36}\equiv\frac{A}{x+6}+\frac{B}{x-6}$, where A and B are constants.

$\frac{x}{x^2-36}\equiv\frac{A(x-6)}{(x+6)(x-6)}+\frac{B(x+6)}{(x+6)(x-6)}$ M1

$\frac{x}{x^2-36}\equiv\frac{Ax-6A+Bx+6B}{(x+6)(x-6)}$

$x\equiv(A+B)x+(-6A+6B)$ A1

$1=A+B$

$B=1-A$

$0=-6A+6B$

$\therefore 0=-6A+6(1-A)$ A1

$12A=6$

$A=\frac{1}{2}$

$\therefore B=1-\frac{1}{2}$

$B=\frac{1}{2}$

$\therefore \frac{x}{x^2-36}\equiv\frac{1}{2(x+6)}+\frac{1}{2(x-6)}$ A1

$f(x)=\frac{1}{2(x+12-6)}+\frac{1}{2(x-6)}$

$f(x)=g(x+12)+g(x)$

$\therefore g(x)=\frac{1}{2(x-6)}$ A1

[5]

4. (a) $x=-5,\ x=5$ A2

[2]

(b) $f'(x)=\frac{(x^2-25)(2x)-(x^2+25)(2x)}{(x^2-25)^2}$ (A1) for correct approach

$f'(x)=\frac{2x^3-50x-2x^3-50x}{(x^2-25)^2}$

$f'(x)=-\frac{100x}{(x^2-25)^2}$ A1

[2]

(c) $f'(x)=0$

$\therefore -\frac{100x}{(x^2-25)^2}=0$ M1

$100x=0$

$x=0$

By the first derivative test, M1A1

x	$x<-5$	$x=-5$	$-5<x<0$	$x=0$	$0<x<5$	$x=5$	$x>5$
$f'(x)$	+	Undefined	+	0	–	Undefined	–

Thus, there is no local minimum of $f(x)$. AG

[3]

(d) $f''(x)=-\frac{(x^2-25)^2(100)-(100x)(2)(x^2-25)(2x)}{((x^2-25)^2)^2}$ (A1) for correct approach

$f''(x)=\frac{-100(x^2-25)^2+400x^2(x^2-25)}{(x^2-25)^4}$

$f''(x)=\frac{-100(x^2-25)+400x^2}{(x^2-25)^3}$ (M1) for simplification

$f''(x)=\frac{-100x^2+2500+400x^2}{(x^2-25)^3}$

$f''(x)=\frac{100(3x^2+25)}{(x^2-25)^3}$ A1

[3]

(e) $f''(x)=0$

$\therefore \dfrac{100(3x^2+25)}{(x^2-25)^3}=0$ M1

$100(3x^2+25)=0$

$3x^2+25=0$

$x^2=-\dfrac{25}{3}$ A1

Therefore, $f''(x)\neq 0$ for $x\neq -5,\ x\neq 5$.

Thus, there is no point of inflexion of $f(x)$. AG

[2]

(f) (i) $f(x)\equiv \dfrac{A}{x+5}+\dfrac{B}{x-5}+C$

$\dfrac{x^2+25}{x^2-25}\equiv \dfrac{A(x-5)}{(x+5)(x-5)}$

$+\dfrac{B(x+5)}{(x+5)(x-5)}+\dfrac{C(x^2-25)}{x^2-25}$ M1A1

$\dfrac{x^2+25}{x^2-25}\equiv \dfrac{Ax-5A+Bx+5B+Cx^2-25C}{(x+5)(x-5)}$

$x^2+25\equiv Cx^2+(A+B)x+(-5A+5B-25C)$ A1

$C=1$ A1

$0=A+B$

$B=-A$

$25=-5A+5B-25C$

$\therefore 25=-5A+5(-A)-25(1)$

$10A=-50$

$A=-5$ A1

$\therefore B=-(-5)$

$B=5$ A1

(ii) $y=1$ A1

[7]

Exercise 47

1.	(a)	$V = \left(\frac{1}{2}r^2\theta\right)h$	A1
		$\therefore 50 = \frac{1}{2}r^2\theta h$	M1
		$h = \frac{100}{r^2\theta}$	AG
			[2]
	(b)	$A = 2\left(\frac{1}{2}r^2\theta\right) + (r\theta)(h) + 2(rh)$	(M1) for valid approach
		$A = r^2\theta + r\theta h + 2rh$	
		$A = r^2\theta + r\theta\left(\frac{100}{r^2\theta}\right) + 2r\left(\frac{100}{r^2\theta}\right)$	(M1) for substitution
		$A = r^2\theta + \frac{100}{r} + \frac{200}{r\theta}$	A1
			[3]

(c) $r = r\theta$

$\theta = 1$ A1

$\therefore A = r^2(1) + \frac{100}{r} + \frac{200}{r(1)}$

$A = r^2 + \frac{300}{r}$ A1

$\frac{dA}{dr} = 2r + 300(-r^{-2})$ A1

$\frac{dA}{dr} = 2r - \frac{300}{r^2}$

$\frac{dA}{dr} = 0$

$\therefore 2r - \frac{300}{r^2} = 0$ M1

$2r^3 - 300 = 0$

$r^3 = 150$

$r = \sqrt[3]{150}$ A1

By the first derivative test, M1A1

x	$0 < x < \sqrt[3]{150}$	$x = \sqrt[3]{150}$	$x > \sqrt[3]{150}$
$\frac{dC}{dx}$	–	0	+

Thus, A attains its minimum when $r = \sqrt[3]{150}$. AG

[7]

(d) $(\sqrt[3]{150})^2 + \frac{300}{\sqrt[3]{150}} = k\sqrt[3]{22500}$ (M1) for setting equation

$\sqrt[3]{150^2} + \frac{300}{\sqrt[3]{150}} = k\sqrt[3]{150^2}$

$150 + 300 = k(150)$ (A1) for correct approach

$450 = 150k$

$k = 3$ A1

[3]

(e) $h = \dfrac{100}{(\sqrt[3]{150})^2(1)}$ (M1) for substitution

$h = \dfrac{100}{\sqrt[3]{150^2}}$

$h = \dfrac{100\sqrt[3]{150}}{150}$ (A1) for correct approach

$h = \dfrac{2}{3}\sqrt[3]{150}$

$h = \dfrac{2}{3}r$

$\therefore b = \dfrac{2}{3}$ A1

[3]

2. (a) $A = 2\pi r^2 + 2\pi rh + \pi r^2$ A1

$A = 3\pi r^2 + 2\pi rh$

$\therefore 125 = 3\pi r^2 + 2\pi rh$ M1

$\frac{125}{2\pi r} = \frac{3}{2}r + h$

$h = \frac{125}{2\pi r} - \frac{3}{2}r$ AG

[2]

(b) $h : r = 11:1$

$\therefore \frac{h}{r} = \frac{11}{1}$

$h = 11r$ (A1) for correct approach

$h = \frac{125}{2\pi r} - \frac{3}{2}r$

$\therefore 11r = \frac{125}{2\pi r} - \frac{3}{2}r$ (M1) for substitution

$11r^2 = \frac{125}{2\pi} - \frac{3}{2}r^2$

$\frac{25}{2}r^2 = \frac{125}{2\pi}$ (A1) for correct approach

$r^2 = \frac{5}{\pi}$

$r = \sqrt{\frac{5}{\pi}}$ A1

[4]

(c) $V = \frac{2}{3}\pi r^3 + \pi r^2 h$ (M1) for valid approach

$V = \frac{2}{3}\pi r^3 + \pi r^2\left(\frac{125}{2\pi r} - \frac{3}{2}r\right)$ (M1) for substitution

$V = \frac{2}{3}\pi r^3 + \frac{125}{2}r - \frac{3}{2}\pi r^3$

$V = \frac{125}{2}r - \frac{5}{6}\pi r^3$ A1

[3]

(d) $V=\frac{125}{2}r-\frac{5}{6}\pi r^3$

$\frac{dV}{dr}=\frac{125}{2}(1)-\frac{5}{6}\pi(3r^2)$ A1

$\frac{dV}{dr}=\frac{125}{2}-\frac{5}{2}\pi r^2$

$\frac{dV}{dr}=0$

$\therefore \frac{125}{2}-\frac{5}{2}\pi r^2=0$ M1

$125-5\pi r^2=0$

$\pi r^2=25$

$r=\sqrt{\frac{25}{\pi}}$

$r=\frac{5}{\sqrt{\pi}}$ A1

By the first derivative test, M1A1

r	$0<r<\frac{5}{\sqrt{\pi}}$	$r=\frac{5}{\sqrt{\pi}}$	$r>\frac{5}{\sqrt{\pi}}$
$\frac{dV}{dr}$	+	0	–

Thus, V attains its maximum when $r=\frac{5}{\sqrt{\pi}}$. AG

[5]

(e) $\frac{125}{2}\left(\frac{5}{\sqrt{\pi}}\right)-\frac{5}{6}\pi\left(\frac{5}{\sqrt{\pi}}\right)^3=\sqrt{\frac{25}{k\pi}}$ (M1) for setting equation

$\frac{625}{2\sqrt{\pi}}-\frac{625}{6\sqrt{\pi}}=\frac{625}{\sqrt{k\pi}}$

$\frac{625}{3\sqrt{\pi}}=\frac{625}{\sqrt{k\pi}}$ (A1) for correct approach

$\because 3\sqrt{\pi}=\sqrt{k\pi}$ (M1) for valid approach

$9\pi=k\pi$

$k=9$ A1

[4]

3. (a) $R^2 = \text{OP}^2 + r^2$ (M1) for valid approach

$\text{OP} = \sqrt{R^2 - r^2}$

$V = \dfrac{1}{3}\pi r^2 h$

$\therefore V = \dfrac{1}{3}\pi r^2 (R + \text{OP})$ (A1) for substitution

$\therefore V = \dfrac{1}{3}\pi r^2 (R + \sqrt{R^2 - r^2})$ A1

[3]

(b) $V = \dfrac{1}{3}\pi r^2 (R + \sqrt{R^2 - r^2})$

$\dfrac{\mathrm{d}V}{\mathrm{d}r} = \dfrac{1}{3}\pi(2r)(R + \sqrt{R^2 - r^2})$

$+\dfrac{1}{3}\pi r^2 \left(\dfrac{1}{2\sqrt{R^2 - r^2}}\right)(-2r)$ M1A1

$\dfrac{\mathrm{d}V}{\mathrm{d}r} = \dfrac{2}{3}\pi r(R + \sqrt{R^2 - r^2}) - \dfrac{1}{3}\pi r^3 \left(\dfrac{1}{\sqrt{R^2 - r^2}}\right)$ A1

$\dfrac{\mathrm{d}V}{\mathrm{d}r} = \dfrac{2\pi r(R\sqrt{R^2 - r^2} + R^2 - r^2)}{3\sqrt{R^2 - r^2}} - \dfrac{\pi r^3}{3\sqrt{R^2 - r^2}}$ M1

$\dfrac{\mathrm{d}V}{\mathrm{d}r} = \dfrac{\pi r(2R\sqrt{R^2 - r^2} + 2R^2 - 2r^2 - r^2)}{3\sqrt{R^2 - r^2}}$ A1

$\dfrac{\mathrm{d}V}{\mathrm{d}r} = \dfrac{\pi r(2R\sqrt{R^2 - r^2} + 2R^2 - 3r^2)}{3\sqrt{R^2 - r^2}}$ AG

[5]

(c) $\frac{dV}{dr}=0$

$\therefore \frac{\pi r(2R\sqrt{R^2-r^2}+2R^2-3r^2)}{3\sqrt{R^2-r^2}}=0$ M1

$2R\sqrt{R^2-r^2}+2R^2-3r^2=0$

$2R\sqrt{R^2-r^2}=3r^2-2R^2$

$\therefore 4R^2(R^2-r^2)=(3r^2-2R^2)^2$ M1

$4R^4-4R^2r^2=9r^4-12R^2r^2+4R^4$ A1

$8R^2r^2=9r^4$

$r^2=\frac{8}{9}R^2$

$r=\frac{2\sqrt{2}}{3}R$ A1

By the first derivative test, M1A1

r	$0<r<\frac{2\sqrt{2}}{3}R$	$r=\frac{2\sqrt{2}}{3}R$	$\frac{2\sqrt{2}}{3}R<r<R$
$\frac{dV}{dr}$	+	0	–

Thus, V attains its maximum when $r=\frac{2\sqrt{2}}{3}R$. AG

[6]

(d) $\frac{1}{3}\pi\left(\frac{8}{9}R^2\right)\left(R+\sqrt{R^2-\frac{8}{9}R^2}\right)=\frac{k}{81}\pi R^3$ (M1) for setting equation

$\frac{8}{27}\pi R^2\left(R+\sqrt{\frac{1}{9}R^2}\right)=\frac{k}{81}\pi R^3$

$\frac{8}{27}\pi R^2\left(R+\frac{1}{3}R\right)=\frac{k}{81}\pi R^3$ (A1) for correct approach

$\frac{8}{27}\pi R^2\left(\frac{4}{3}R\right)=\frac{k}{81}\pi R^3$

$\frac{32}{81}\pi R^3=\frac{k}{81}\pi R^3$ (A1) for correct approach

$k=32$ A1

[4]

(e) The minimum capacity

$= \frac{4}{3}\pi R^3 - \frac{32}{81}\pi R^3$ (M1) for valid approach

$= \frac{76}{81}\pi R^3$ A1

[2]

4. (a) Let l be the distance between the centre of the top square face and one of its vertices.

$l^2 + l^2 = x^2$ M1

$2l^2 = x^2$

$\sqrt{2}l = x$

$l = \dfrac{x}{\sqrt{2}}$ A1

By considering a pair of similar triangles,

$\dfrac{H-y}{l} = \dfrac{H}{R}$ M1

$H - y = \dfrac{H}{R}l$

$y = H - \dfrac{H}{R}l$

$y = H - \dfrac{H}{R}\left(\dfrac{x}{\sqrt{2}}\right)$ M1

$y = H - \dfrac{H}{\sqrt{2}R}x$

$y = H - \dfrac{\sqrt{2}H}{2R}x$ AG

[4]

(b) $V = x^2 y$

$\therefore V = x^2\left(H - \dfrac{\sqrt{2}H}{2R}x\right)$ (M1) for substitution

$V = Hx^2 - \dfrac{\sqrt{2}H}{2R}x^3$ A1

[2]

(c) $\frac{dV}{dx}=H(2x)-\frac{\sqrt{2}H}{2R}(3x^2)$ A1

$\frac{dV}{dx}=2Hx-\frac{3\sqrt{2}H}{2R}x^2$ A1

$\frac{dV}{dx}=0$

$\therefore 2Hx-\frac{3\sqrt{2}H}{2R}x^2=0$ M1

$4HRx-3\sqrt{2}Hx^2=0$

$4HRx=3\sqrt{2}Hx^2$

$x=\frac{4}{3\sqrt{2}}R$

$x=\frac{2\sqrt{2}}{3}R$ A1

By the first derivative test, M1A1

x	$0<x<\frac{2\sqrt{2}}{3}R$	$x=\frac{2\sqrt{2}}{3}R$	$\frac{2\sqrt{2}}{3}R<x<R$
$\frac{dV}{dx}$	+	0	–

Thus, V attains its maximum when $x=\frac{2\sqrt{2}}{3}R$. AG

[6]

(d) The maximum value of V

$=H\left(\frac{2\sqrt{2}}{3}R\right)^2-\frac{\sqrt{2}H}{2R}\left(\frac{2\sqrt{2}}{3}R\right)^3$ (M1) for substitution

$=H\left(\frac{8}{9}R^2\right)-\frac{\sqrt{2}H}{2R}\left(\frac{16\sqrt{2}}{27}R^3\right)$ (A1) for correct approach

$=\frac{8}{9}HR^2-\frac{16}{27}HR^2$

$=\frac{8}{27}HR^2$ A1

[3]

(e) The total surface area

$= 2x^2 + 4xy$ (M1) for valid approach

$= 2\left(\frac{2\sqrt{2}}{3}R\right)^2 + 4\left(\frac{2\sqrt{2}}{3}R\right)\left(H - \frac{\sqrt{2}H}{2R}\left(\frac{2\sqrt{2}}{3}R\right)\right)$ (A1) for substitution

$= 2\left(\frac{8}{9}R^2\right) + 4\left(\frac{2\sqrt{2}}{3}R\right)\left(H - \frac{2H}{3}\right)$ (A1) for correct approach

$= \frac{16}{9}R^2 + \frac{4H}{3}\left(\frac{2\sqrt{2}}{3}R\right)$

$= \frac{16}{9}R^2 + \frac{8\sqrt{2}}{9}HR$

$= \frac{8}{9}R(2R + \sqrt{2}H)$ A1

[4]

Exercise 48

1. (a) $x^2 + y^2 = 50^2$

$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(2500)$ (M1) for valid approach

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ (A1) for correct approach

$x^2 + 14^2 = 50^2$

$x^2 = 2304$

$x = 48$

$\therefore 2(48)\frac{dx}{dt} + 2(14)(10) = 0$ A1

$96\frac{dx}{dt} = -280$

$\frac{dx}{dt} = -\frac{35}{12}\text{ ms}^{-1}$ A1

[4]

(b) Let $\theta = \text{O}\hat{\text{B}}\text{A}$.

$\sin\theta = \frac{y}{50}$

$\frac{d}{dt}(\sin\theta) = \frac{d}{dt}\left(\frac{y}{50}\right)$

$\cos\theta\frac{d\theta}{dt} = \frac{1}{50}\frac{dy}{dt}$ (A1) for correct approach

$\sin\theta = \frac{14}{50}$

$\therefore \sqrt{1 - \left(\frac{14}{50}\right)^2}\frac{d\theta}{dt} = \frac{1}{50}(10)$ (A1) for substitution

$\frac{24}{25}\frac{d\theta}{dt} = \frac{1}{5}$

$\frac{d\theta}{dt} = \frac{5}{24}\text{ rads}^{-1}$ A1

[3]

2. (a) $x^2 + y^2 = H^2$ A1

$\frac{\mathrm{d}}{\mathrm{d}t}(x^2) + \frac{\mathrm{d}}{\mathrm{d}t}(y^2) = \frac{\mathrm{d}}{\mathrm{d}t}(H^2)$ M1

$2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 2H\frac{\mathrm{d}H}{\mathrm{d}t}$ A1

$\therefore 2x(40) + 2y(9) = 2\sqrt{x^2 + y^2}\frac{\mathrm{d}H}{\mathrm{d}t}$ A1

$40x + 9y = \sqrt{x^2 + y^2}\frac{\mathrm{d}H}{\mathrm{d}t}$

$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{40x + 9y}{\sqrt{x^2 + y^2}}$ AG

[4]

(b) $40 = \frac{x}{t} \Rightarrow x = 40t$ (M1) for valid approach

$9 = \frac{y}{t} \Rightarrow y = 9t$

$\therefore \frac{\mathrm{d}H}{\mathrm{d}t} = \frac{40(40t) + 9(9t)}{\sqrt{(40t)^2 + (9t)^2}}$ (A1) for correct approach

$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{1681t}{41t}$

$\frac{\mathrm{d}H}{\mathrm{d}t} = 41$ A1

[3]

3. $z = xy^2 + \ln y$

$\frac{\mathrm{d}}{\mathrm{d}t}(z) = \frac{\mathrm{d}}{\mathrm{d}t}(xy^2) + \frac{\mathrm{d}}{\mathrm{d}t}(\ln y)$ (M1) for valid approach

$\frac{\mathrm{d}z}{\mathrm{d}t} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)(y^2) + (x)\left(2y\frac{\mathrm{d}y}{\mathrm{d}t}\right) + \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t}$ (A2) for correct approach

$\frac{\mathrm{d}z}{\mathrm{d}t} = y^2\frac{\mathrm{d}x}{\mathrm{d}t} + \left(2xy + \frac{1}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}t}$

$5 = xe^2 + \ln e$ (M1) for substitution

$4 = xe^2$

$x = \frac{4}{e^2}$

$\therefore \frac{\mathrm{d}z}{\mathrm{d}t} = e^2\left(-\frac{5}{e}\right) + \left(2\left(\frac{4}{e^2}\right)e + \frac{1}{e}\right)(e^2)$ (A1) for substitution

$\frac{\mathrm{d}z}{\mathrm{d}t} = -5e + \left(\frac{9}{e}\right)(e^2)$

$\frac{\mathrm{d}z}{\mathrm{d}t} = 4e$ A1

[6]

4. $\dfrac{\sin\beta}{x} = \dfrac{\sin\frac{\pi}{6}}{60}$ (M1)(A1) for substitution

$\dfrac{\sin\beta}{x} = \dfrac{1}{120}$

$x = 120\sin\beta$ A1

$\dfrac{\mathrm{d}}{\mathrm{d}t}(x) = \dfrac{\mathrm{d}}{\mathrm{d}t}(120\sin\beta)$

$\dfrac{\mathrm{d}x}{\mathrm{d}t} = 120\cos\beta\dfrac{\mathrm{d}\beta}{\mathrm{d}t}$ A1

$\dfrac{\sin\beta}{60\sqrt{2}} = \dfrac{\sin\frac{\pi}{6}}{60}$

$\sin\beta = \dfrac{\sqrt{2}}{2}$

$\beta = \dfrac{\pi}{4}$ (A1) for correct value

$\therefore 90\sqrt{2} = 120\cos\dfrac{\pi}{4}\dfrac{\mathrm{d}\beta}{\mathrm{d}t}$ (A1) for substitution

$90\sqrt{2} = 120\left(\dfrac{\sqrt{2}}{2}\right)\dfrac{\mathrm{d}\beta}{\mathrm{d}t}$

$90\sqrt{2} = 60\sqrt{2}\dfrac{\mathrm{d}\beta}{\mathrm{d}t}$

$\dfrac{\mathrm{d}\beta}{\mathrm{d}t} = 1.5\ \text{rads}^{-1}$ A1

[7]

Exercise 49

1. $s^2+4s+5t^4=0$

$\frac{d}{dt}(s^2)+\frac{d}{dt}(4s)+\frac{d}{dt}(5t^4)=\frac{d}{dt}(0)$ (M1) for valid approach

$2s\frac{ds}{dt}+4\frac{ds}{dt}+20t^3=0$ (A1) for correct approach

$(2s+4)\frac{ds}{dt}=-20t^3$

$\frac{ds}{dt}=-\frac{20t^3}{2s+4}$

$\frac{ds}{dt}=-\frac{10t^3}{s+2}$ A1

$\frac{d^2s}{dt^2}=-\frac{(s+2)\frac{d}{dt}(10t^3)-(10t^3)\frac{d}{dt}(s+2)}{(s+2)^2}$ (A1) for correct approach

$\frac{d^2s}{dt^2}=-\frac{(s+2)(30t^2)-10t^3\frac{ds}{dt}}{(s+2)^2}$

$\frac{d^2s}{dt^2}=-\frac{30t^2}{s+2}+\frac{10t^3}{(s+2)^2}\left(-\frac{10t^3}{s+2}\right)$ (M1) for substitution

$\frac{d^2s}{dt^2}=-\frac{30t^2}{s+2}-\frac{100t^6}{(s+2)^3}$ A1

[6]

2. $s = \tan^{-1} t^2$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \left(\frac{1}{1+(t^2)^2}\right)(2t)$$ (A1) for correct approach

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{2t}{1+t^4}$$

$$\frac{\mathrm{d}^2s}{\mathrm{d}t^2} = \frac{(1+t^4)(2)-(2t)(4t^3)}{(1+t^4)^2}$$ (A1) for correct approach

$$\frac{\mathrm{d}^2s}{\mathrm{d}t^2} = \frac{2+2t^4-8t^4}{(1+t^4)^2}$$

$$\frac{\mathrm{d}^2s}{\mathrm{d}t^2} = \frac{2-6t^4}{(1+t^4)^2}$$ A1

$$\therefore \left.\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\right|_{t=\sqrt{2}} = \frac{2-6(\sqrt{2})^4}{(1+(\sqrt{2})^4)^2}$$ (M1) for substitution

$$\left.\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\right|_{t=\sqrt{2}} = -\frac{22}{25}\ \mathrm{ms}^{-2}$$ A1

[5]

3. $s = \log_4(3-2e^t)$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \left(\frac{1}{(3-2e^t)\ln 4}\right)(-2e^t)$$ A1

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -\frac{2e^t}{(3-2e^t)\ln 4}$$

$$\frac{\mathrm{d}^2s}{\mathrm{d}t^2} = -\frac{1}{\ln 4}\left[\frac{(3-2e^t)(2e^t)-(2e^t)(-2e^t)}{(3-2e^t)^2}\right]$$ A1

$$\frac{\mathrm{d}^2s}{\mathrm{d}t^2} = -\frac{1}{\ln 4}\left[\frac{6e^t-4e^{2t}+4e^{2t}}{(3-2e^t)^2}\right]$$ A1

$$\frac{\mathrm{d}^2s}{\mathrm{d}t^2} = -\frac{6e^t}{(3-2e^t)^2\ln 4}$$ A1

$$\therefore \left.\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\right|_{t=\ln 2} = -\frac{6e^{\ln 2}}{(3-2e^{\ln 2})^2\ln 4}$$ M1

$$\left.\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\right|_{t=\ln 2} = -\frac{6(2)}{(3-2(2))^2\ln 4}$$

$$\left.\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\right|_{t=\ln 2} = -\frac{12}{\ln 4}\ \mathrm{ms}^{-2}$$ AG

[5]

4. (a) $s = t\arccos t$

$$\frac{ds}{dt} = (1)(\arccos t) + (t)\left(-\frac{1}{\sqrt{1-t^2}}\right)$$ (A1) for correct approach

$$\frac{ds}{dt} = \arccos t - \frac{t}{\sqrt{1-t^2}}$$

$$\frac{ds}{dt} = 0$$ (M1) for setting equation

By considering the graph of $y = \arccos t - \frac{t}{\sqrt{1-t^2}}$,

$t = 0.6521846$.

$\therefore t = 0.652$ A1

[3]

(b) $$\frac{d^2s}{dt^2} = -\frac{1}{\sqrt{1-t^2}} - \frac{(\sqrt{1-t^2})(1) - (t)\left(\frac{1}{2\sqrt{1-t^2}}\right)(-2t)}{(\sqrt{1-t^2})^2}$$ (A1) for correct approach

$$\frac{d^2s}{dt^2} = -\frac{1}{\sqrt{1-t^2}} - \frac{\sqrt{1-t^2} + \frac{t^2}{\sqrt{1-t^2}}}{1-t^2}$$

$$\frac{d^2s}{dt^2} = -\frac{1-t^2}{(1-t^2)^{\frac{3}{2}}} - \frac{1-t^2}{(1-t^2)^{\frac{3}{2}}} - \frac{t^2}{(1-t^2)^{\frac{3}{2}}}$$ (M1) for valid approach

$$\frac{d^2s}{dt^2} = \frac{-2+t^2}{(1-t^2)^{\frac{3}{2}}}$$

$\therefore k = -2$ A1

[3]

Exercise 50

1. (a) The triangles ABP and PCD are similar.

$\therefore \frac{\text{CD}}{x} = \frac{11-x}{4}$ (M1) for valid approach

$\text{CD} = \frac{1}{4}x(11-x)$

$\text{CD} = -\frac{1}{4}x^2 + \frac{11}{4}x$ A1

[2]

(b) (i) $H = \frac{(4)(x)}{2} + \frac{(11-x)\left(-\frac{1}{4}x^2 + \frac{11}{4}x\right)}{2}$ (M1) for valid approach

$H = 2x + (11-x)\left(-\frac{1}{8}x^2 + \frac{11}{8}x\right)$

$H = 2x - \frac{11}{8}x^2 + \frac{121}{8}x + \frac{1}{8}x^3 - \frac{11}{8}x^2$ (A1) for correct approach

$H = \frac{1}{8}x^3 - \frac{11}{4}x^2 + \frac{137}{8}x$ A1

(ii) $\frac{dH}{dx} = \frac{1}{8}(3x^2) - \frac{11}{4}(2x) + \frac{137}{8}(1)$ (A1) for correct approach

$\frac{dH}{dx} = \frac{3}{8}x^2 - \frac{11}{2}x + \frac{137}{8}$

$\frac{dH}{dx} = 0$

$\therefore \frac{3}{8}x^2 - \frac{11}{2}x + \frac{137}{8} = 0$ (M1) for setting equation

$3x^2 - 44x + 137 = 0$

By considering the graph of

$y = 3x^2 - 44x + 137$, $x = 4.4853321$ or

$x = 10.181335$.

By the first derivative test, M1A1

x	$0 < x < 4.4853321$	$x = 4.4853321$	$4.4853321 < x < 10.181335$
dH/dx	$+$	0	$-$
x	$x = 10.181335$	$10.181335 < x < 11$	$x = 11$
dH/dx	0	$+$	$+$

Thus, H attains its local maximum at

$x = 4.4853321$. R1

When $x = 4.4853321$,

$H = \frac{1}{8}(4.4853321)^3 - \frac{11}{4}(4.4853321)^2$

$+\frac{137}{8}(4.4853321)$ (M1) for susbtitution

$H = 32.76585438$

When $x = 11$,

$H = \frac{1}{8}(11)^3 - \frac{11}{4}(11)^2 + \frac{137}{8}(11)$

$H = 22$

Hence, the maximum value of H is 32.8. A1

(iii) $x = 4.49$ A1

[11]

(c) $\tan\theta = \frac{4}{x}$

$\sec^2\theta \frac{d\theta}{dx} = -\frac{4}{x^2}$ (A1) for correct approach

$\frac{d\theta}{dx} = -\frac{4\cos^2\theta}{x^2}$

$\frac{d\theta}{dx} = -\frac{4}{x^2}\left(\frac{x}{\sqrt{4^2 + x^2}}\right)^2$ (M1) for valid approach

$\frac{d\theta}{dx} = -\frac{4}{16 + x^2}$ A1

[3]

(d) $\frac{dH}{dx} = \frac{dH}{d\theta} \cdot \frac{d\theta}{dx}$ (M1) for valid approach

$\frac{3}{8}x^2 - \frac{11}{2}x + \frac{137}{8} = \frac{dH}{d\theta}\left(-\frac{4}{16 + x^2}\right)$ (A1) for substitution

$\frac{3x^2 - 44x + 137}{8} = \frac{dH}{d\theta}\left(-\frac{4}{16 + x^2}\right)$

$\frac{dH}{d\theta} = -\frac{(3x^2 - 44x + 137)(16 + x^2)}{32}$ A1

[3]

2. (a) $K = 2\left(\frac{1}{2}(x-1)(6-x)^2 \sin 60^\circ\right)$ (A1) for substitution

$K = (x-1)(36-12x+x^2)\left(\frac{\sqrt{3}}{2}\right)$

$K = \frac{\sqrt{3}}{2}(-36+48x-13x^2+x^3)$ A1

[2]

(b) (i) $\frac{dK}{dx} = \frac{\sqrt{3}}{2}(0+48(1)-13(2x)+3x^2)$ (A1) for correct approach

$\frac{dK}{dx} = \frac{\sqrt{3}}{2}(48-26x+3x^2)$ A1

(ii) $\frac{dK}{dx} = 0$

$\therefore \frac{\sqrt{3}}{2}(48-26x+3x^2) = 0$ (M1) for setting equation

$3x^2 - 26x + 48 = 0$

$(3x-8)(x-6) = 0$

$x = \frac{8}{3}$ or $x = 6$ (*Rejected*)

By the first derivative test, M1A1

x	$0 < x < \frac{8}{3}$	$x = \frac{8}{3}$	$\frac{8}{3} < x < 6$
$\frac{dK}{dx}$	+	0	−

Thus, K attains its maximum at $x = \frac{8}{3}$. R1

The maximum value of K

$= \frac{\sqrt{3}}{2}\left(-36+48\left(\frac{8}{3}\right)-13\left(\frac{8}{3}\right)^2+\left(\frac{8}{3}\right)^3\right)$

$= \frac{250\sqrt{3}}{27}$ A1

(iii) $x = \frac{8}{3}$ A1

[8]

(c) $BC^2 = BD^2 + CD^2 - 2(BD)(CD)\cos B\hat{D}C$ (M1) for cosine rule

$BC^2 = (x-1)^2 + (6-x)^4 - 2(x-1)(6-x)^2 \cos 60^\circ$ (A1) for substitution

$BC^2 = (x-1)^2 + (6-x)^4 - (x-1)(6-x)^2$

$BC = \sqrt{(x-1)^2 + (6-x)^4 - (x-1)(6-x)^2}$

By considering the graph of

$y = \sqrt{(x-1)^2 + (6-x)^4 - (x-1)(6-x)^2}$,

$3.0634008 \le y < 25$.

Thus, the range of values of BC is $3.06 \le BC < 25$. A2

[4]

(d) When K attains its maximum, $x = \frac{8}{3}$.

$BC = \sqrt{\left(\frac{8}{3}-1\right)^2 + \left(6-\frac{8}{3}\right)^4 - \left(\frac{8}{3}-1\right)\left(6-\frac{8}{3}\right)^2}$ (M1) for substitution

$BC = 10.378634$ A1

$BC < 25$

Therefore, BC does not attain its maximum when K attains its maximum.

Thus, the claim is disagreed. A1

[3]

3. (a) $DP^2 + 250^2 = x^2$

$DP = \sqrt{x^2 - 62500}$ (A1) for correct approach

$T = \frac{AP}{2} + \frac{PC}{4}$ (M1) for valid approach

$T = \frac{x}{2} + \frac{500 - \sqrt{x^2 - 62500}}{4}$

$T = \frac{2x + 500 - \sqrt{x^2 - 62500}}{4}$ A1

[3]

(b) (i) $\frac{dT}{dx} = \frac{1}{4}\left(2(1) + 0 - \frac{1}{2}(x^2 - 62500)^{-\frac{1}{2}}(2x - 0)\right)$ (A1) for correct approach

$\frac{dT}{dx} = \frac{1}{4}\left(2 - \frac{x}{\sqrt{x^2 - 62500}}\right)$

$\frac{dT}{dx} = \frac{2\sqrt{x^2 - 62500} - x}{4\sqrt{x^2 - 62500}}$ A1

(ii) $\frac{dT}{dx} = 0$

$\therefore \frac{2\sqrt{x^2 - 62500} - x}{4\sqrt{x^2 - 62500}} = 0$ (M1) for setting equation

$2\sqrt{x^2 - 62500} - x = 0$

$2\sqrt{x^2 - 62500} = x$

$4(x^2 - 62500) = x^2$

$4x^2 - 250000 = x^2$ (A1) for correct approach

$3x^2 = 250000$

$x^2 = \frac{250000}{3}$

$x = \frac{500\sqrt{3}}{3}$ or $x = -\frac{500\sqrt{3}}{3}$ (*Rejected*)

By the first derivative test, M1A1

x	$250 < x < \frac{500\sqrt{3}}{3}$	$x = \frac{500\sqrt{3}}{3}$	$x > \frac{500\sqrt{3}}{3}$
$\frac{dT}{dx}$	$-$	0	$+$

Thus, T attains its minimum at $x = \frac{500\sqrt{3}}{3}$. R1

The minimum value of T

$$= \frac{2\left(\frac{500\sqrt{3}}{3}\right) + 500 - \sqrt{\left(\frac{500\sqrt{3}}{3}\right)^2 - 62500}}{4}$$

$= 233.2531755$

$= 233\text{ s}$ A1

(iii) $x = \frac{500\sqrt{3}}{3}$ A1

[9]

(c) $L = \text{AP} + \text{PC}$

$L = x + 500 - \sqrt{x^2 - 62500}$ A1

$\frac{dL}{dx} = 1 + 0 - \frac{1}{2}(x^2 - 62500)^{-\frac{1}{2}}(2x - 0)$ A1

$\frac{dL}{dx} = 1 - \frac{x}{\sqrt{x^2 - 62500}}$ AG

[2]

(d) When T attains its minimum, $x = \frac{500\sqrt{3}}{3}$.

By considering the graph of

$y = x + 500 - \sqrt{x^2 - 62500}$, the graph of L is

concave upward. (M1) for valid approach

$\therefore \frac{d^2L}{dx^2} > 0$ for $x \geq 250$

$\frac{d}{dx}\left(\frac{dL}{dx}\right) > 0$ for $x \geq 250$ R1

Therefore, $\frac{dL}{dx}$ is increasing when T attains its

minimum.

Thus, the claim is agreed. A1

[3]

4. (a) $g(x) = e^{-x^2}$

$g'(x) = (e^{-x^2})(-2x)$ (A1) for correct approach

$g'(x) = -2xe^{-x^2}$

$g''(x) = (-2)(e^{-x^2}) + (-2x)(e^{-x^2})(-2x)$ (A1) for correct approach

$g''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$

$g''(x) = 2e^{-x^2}(2x^2 - 1)$ A1

[3]

(b) $0 < x < \frac{1}{\sqrt{2}}$

$0 < x^2 < \frac{1}{2}$

$0 < 2x^2 < 1$

$-1 < 2x^2 - 1 < 0$ A1

$e^{-x^2} > 0$ for all values of x. A1

$\therefore g''(x) < 0$ for $0 < x < \frac{1}{\sqrt{2}}$.

Thus, the graph of $g(x)$ is concave downward for

$0 < x < \frac{1}{\sqrt{2}}$. AG

[2]

(c) (i) $\text{OA} = e^{-0^2} - 0$

$\text{OA} = 1$

$\text{BB}' = e^{-h^2} - 0$

$\text{BB}' = e^{-h^2}$ (A1) for correct value

$\text{CC}' = e^{-\left(\frac{1}{\sqrt{2}}\right)^2} - 0$

$\text{CC}' = e^{-\frac{1}{2}}$ (A1) for correct value

$T = \frac{(1+e^{-h^2})(h)}{2} + \frac{(e^{-h^2} + e^{-\frac{1}{2}})\left(\frac{1}{\sqrt{2}} - h\right)}{2}$ (A1) for correct approach

$T = \frac{h + he^{-h^2}}{2}$

$+ \frac{\frac{1}{\sqrt{2}}e^{-h^2} + \frac{1}{\sqrt{2}}e^{-\frac{1}{2}} - he^{-h^2} - he^{-\frac{1}{2}}}{2}$

$$T = \frac{2h + \sqrt{2}e^{-h^2} + \sqrt{2}e^{-\frac{1}{2}} - 2he^{-\frac{1}{2}}}{4}$$ A1

(ii) $\frac{dT}{dh} = \frac{1}{4}\left(2(1) + \sqrt{2}e^{-h^2}(-2h) + 0 - 2e^{-\frac{1}{2}}(1)\right)$ (A1) for correct approach

$\frac{dT}{dh} = \frac{1}{4}\left(2 - 2\sqrt{2}he^{-h^2} - 2e^{-\frac{1}{2}}\right)$

$\frac{dT}{dh} = \frac{1}{2}\left(1 - \sqrt{2}he^{-h^2} - e^{-\frac{1}{2}}\right)$ A1

(iii) $\frac{dT}{dh} = 0$

$\therefore \frac{1}{2}\left(1 - \sqrt{2}he^{-h^2} - e^{-\frac{1}{2}}\right) = 0$ (M1) for setting equation

$1 - \sqrt{2}he^{-h^2} - e^{-\frac{1}{2}} = 0$

By considering the graph of

$y = 1 - \sqrt{2}he^{-h^2} - e^{-\frac{1}{2}}$, $h = 0.3054287$.

By the first derivative test, M1A1

h	$h = 0$	$0 < h < 0.3054287$	$h = 0.3054287$	$0.3054287 < h < \frac{1}{\sqrt{2}}$	$h = \frac{1}{\sqrt{2}}$
$\frac{dT}{dh}$	+	+	0	–	–

Thus, T attains its maximum at

$h = 0.3054287$. R1

The maximum value of T

$$= \frac{2(0.3054287) + \sqrt{2}e^{-0.3054287^2} + \sqrt{2}e^{-\frac{1}{2}} - 2(0.3054287)e^{-\frac{1}{2}}}{4}$$

$= 0.5965926$

$= 0.597$ A1

(iv) $h = 0.3054287$ A1

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(d) $T \geq \frac{2(0)+\sqrt{2}e^{-0^2}+\sqrt{2}e^{-\frac{1}{2}}-2(0)e^{-\frac{1}{2}}}{4}$ M1

$T \geq \frac{\sqrt{2}+\sqrt{2}e^{-\frac{1}{2}}}{4}$ A1

$T \geq \frac{\sqrt{2}\left(1+e^{-\frac{1}{2}}\right)}{4}$

$T \geq \frac{\sqrt{2}}{4}\left(1+\frac{1}{\sqrt{e}}\right)$ AG

[2]