AA HL Practice Set 3 Paper 1 Solution

Section A

1. (a) The common difference

$=95-100$ (M1) for valid approach

$=-5$ A1

[2]

(b) The fifteenth term

$=100+(15-1)(-5)$ (A1) for substitution

$=30$ A1

[2]

(c) The sum of the first fifteen terms

$=\frac{15}{2}\left[2(100)+(15-1)(-5)\right]$ (A1) for substitution

$=975$ A1

[2]

2. (a) The gradient of L_1 is 2. A1

The y-intercept of L_1 is -20. A1

[2]

(b) The gradient of L_2 is $-\frac{1}{2}$. (A1) for correct value

The equation of L_2:

$y-(-20)=-\frac{1}{2}(x-0)$ A1

$y+20=-\frac{1}{2}x$

$2y+40=-x$

$x+2y+40=0$ A1

[3]

3. (a) (i) 4 A1

(ii) $\frac{1}{3}$ A1

(iii) -1 A1

[3]

(b) $\log_{27} x + \frac{8}{3} = \log_4 256 + \log_{125} 5 + \log_\pi \frac{1}{\pi}$

$\log_{27} x + \frac{8}{3} = 4 + \frac{1}{3} - 1$ (M1) for substitution

$\log_{27} x = \frac{2}{3}$

$x = 27^{\frac{2}{3}}$ (A1) for correct approach

$x = (3^3)^{\frac{2}{3}}$

$x = 3^2$

$x = 9$ A1

[3]

4. $\left(1 - \frac{3}{4}x\right)^n (1 + 2nx)^3$

$= \left(1 + \binom{n}{1}\left(-\frac{3}{4}x\right) + \cdots\right)\left(1 + \binom{3}{1}(2nx) + \cdots\right)$ (M1) for valid expansion

$= \left(1 + (n)\left(-\frac{3}{4}x\right) + \cdots\right)(1 + (3)(2nx) + \cdots)$ (A1) for correct approach

$= \left(1 - \frac{3}{4}nx + \cdots\right)(1 + 6nx + \cdots)$ A2

The coefficient of x

$= (1)(6n) + \left(-\frac{3}{4}n\right)(1)$ (A1) for correct approach

$= \frac{21}{4}n$

$\therefore \frac{21}{4}n = \frac{105}{4}$ (M1) for setting equation

$n = 5$ A1

[7]

5. $-3\sqrt{3} \le f(x) \le 3\sqrt{3}$

$-3\sqrt{3} \le 6\sin 2x \le 3\sqrt{3}$

$-\frac{\sqrt{3}}{2} \le \sin 2x \le \frac{\sqrt{3}}{2}$ A1

$\therefore \sin\left(-\frac{\pi}{3}\right) \le \sin 2x \le \sin\frac{\pi}{3}$,

$\sin\left(\pi - \frac{\pi}{3}\right) \le \sin 2x \le \sin\left(\pi + \frac{\pi}{3}\right)$ or (A2) for correct ranges

$\sin\left(2\pi - \frac{\pi}{3}\right) \le \sin 2x \le \sin\left(2\pi + \frac{\pi}{3}\right)$

$-\frac{\pi}{3} \le 2x \le \frac{\pi}{3}$, $\frac{2\pi}{3} \le 2x \le \frac{4\pi}{3}$ or $\frac{5\pi}{3} \le 2x \le \frac{7\pi}{3}$ A1

$-\frac{\pi}{6} \le x \le \frac{\pi}{6}$, $\frac{\pi}{3} \le x \le \frac{2\pi}{3}$ or $\frac{5\pi}{6} \le x \le \frac{7\pi}{6}$ (M1) for valid approach

$\therefore 0 \le x \le \frac{\pi}{6}$, $\frac{\pi}{3} \le x \le \frac{2\pi}{3}$ or $\frac{5\pi}{6} \le x \le \frac{7\pi}{6}$ A3

[8]

6. (a) $E(X)=\int_{-2}^{3} x\cdot\frac{1}{5}\,dx$ (A1) for substitution

$E(X)=\left[\frac{1}{10}x^2\right]_{-2}^{3}$

$E(X)=\frac{9}{10}-\frac{4}{10}$

$E(X)=\frac{1}{2}$ A1

[2]

(b) $E(X^2)=\int_{-2}^{3} x^2\cdot\frac{1}{5}\,dx$ (A1) for substitution

$E(X^2)=\left[\frac{1}{15}x^3\right]_{-2}^{3}$

$E(X^2)=\frac{27}{15}-\left(-\frac{8}{15}\right)$

$E(X^2)=\frac{7}{3}$ A1

[2]

(c) Standard deviation

$=\sqrt{E(X^2)-(E(X))^2}$ (A1) for substitution

$=\sqrt{\frac{7}{3}-\left(\frac{1}{2}\right)^2}$

$=\sqrt{\frac{25}{12}}$ A1

[2]

7. (a) $f(|-x|)=\frac{7-2|-x|}{3-|-x|}$ M1

$f(|-x|)=\frac{7-2|x|}{3-|x|}$ A1

$f(|-x|)=f(|x|)$

Thus, $f(|x|)$ is an even function. AG

[2]

(b) $x=3$, $x=-3$ A2

[2]

(c) $y=-2$ A1

[1]

8.

$$2(\sec\alpha+2\tan\alpha)^2=3+8\sec\alpha\tan\alpha+6\tan^2\alpha$$

$$2(\sec^2\alpha+4\sec\alpha\tan\alpha+4\tan^2\alpha)$$
$$=3+8\sec\alpha\tan\alpha+6\tan^2\alpha$$
(M1) for valid approach

$$2\sec^2\alpha+8\sec\alpha\tan\alpha+8\tan^2\alpha$$
$$=3+8\sec\alpha\tan\alpha+6\tan^2\alpha$$

$$2\sec^2\alpha+2\tan^2\alpha=3$$

$$\sec^2\alpha+\tan^2\alpha=\frac{3}{2}$$

$$1+\tan^2\alpha+\tan^2\alpha=\frac{3}{2}$$
A1

$$2\tan^2\alpha=\frac{1}{2}$$

$$\tan^2\alpha=\frac{1}{4}$$

$\tan\alpha=-\frac{1}{2}$ or $\tan\alpha=\frac{1}{2}$ (*Rejected*) (A1) for correct value

$$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}$$
(M1) for valid approach

$$\tan 2\alpha=\frac{2\left(-\frac{1}{2}\right)}{1-\left(-\frac{1}{2}\right)^2}$$

$$\tan 2\alpha=-\frac{4}{3}$$
A1

[5]

9. When $n=1$,

$1^3+3(1)^2-1=3$

$1^3+3(1)^2-1=3(1)$ A1

Thus, the statement is true when $n=1$.

Assume that the statement is true when $n=k$. M1

$k^3+3k^2-k=3M$, where $M\in\mathbb{Z}$.

When $n=k+1$,

$(k+1)^3+3(k+1)^2-(k+1)$

$=k^3+3k^2+3k+1+3(k^2+2k+1)-k-1$ M1

$=(3M+k-3k^2)+3k^2+3k+1+3k^2+6k+3-k-1$ A1

$=3M+3k^2+9k+3$ M1

$=3(M+k^2+3k+1)$, where $M+k^2+3k+1\in\mathbb{Z}$. A1

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$. R1

[7]

Section B

10. (a) $g(x) - f(x) = 0$

$$e^{\frac{1}{2}\sqrt{x}} - e^{\frac{1}{2}\sqrt{x}} \sin\left(\frac{\pi}{3}x\right) = 0$$ (M1) for valid approach

$$e^{\frac{1}{2}\sqrt{x}}\left(1 - \sin\left(\frac{\pi}{3}x\right)\right) = 0$$

$$1 - \sin\left(\frac{\pi}{3}x\right) = 0$$

$$\sin\left(\frac{\pi}{3}x\right) = 1$$ A1

$$\frac{\pi}{3}x = \frac{\pi}{2}, \ \frac{\pi}{3}x = \frac{5\pi}{2} \text{ or } \frac{\pi}{3}x = \frac{9\pi}{2}$$ (A1) for correct values

$$x = \frac{3}{2}, \ x = \frac{15}{2} \text{ or } x = \frac{27}{2}$$ A3

[6]

(b) (i) $\frac{\pi}{3}x_n = \frac{\pi}{2} + (n-1)(2\pi)$ A1

$$x_n = \frac{3}{2} + 6(n-1)$$

$$x_{n+1} - x_n$$

$$= \left(\frac{3}{2} + 6((n+1)-1)\right) - \left(\frac{3}{2} + 6(n-1)\right)$$ M1

$$x_{n+1} - x_n = \left(\frac{3}{2} + 6n\right) - \left(\frac{3}{2} + 6n - 6\right)$$

$$x_{n+1} - x_n = 6$$ A1

The differences between each pair of consecutive terms are equal to 6.
Thus, $x_1, x_2, x_3, \ldots$ is an arithmetic sequence. AG

(ii) $x_n = \frac{3}{2} + 6n - 6$

$$x_n = 6n - \frac{9}{2}$$ A1

[4]

(c) Note that $x_2 = \frac{15}{2}$ and $x_3 = \frac{27}{2}$.

$f(x) = 0$

$e^{\frac{1}{2}\sqrt{x}} \sin\left(\frac{\pi}{3}x\right) = 0$ M1

$\sin\left(\frac{\pi}{3}x\right) = 0$

$\frac{\pi}{3}x = 3\pi$ or $\frac{\pi}{3}x = 4\pi$

$x = 9$ or $x = 12$ (A1) for correct values

$\therefore R = \int_{\frac{15}{2}}^{9}\left(e^{\frac{1}{2}\sqrt{x}} - e^{\frac{1}{2}\sqrt{x}} \sin\left(\frac{\pi}{3}x\right)\right)\mathrm{d}x + \int_{9}^{12} e^{\frac{1}{2}\sqrt{x}}\,\mathrm{d}x$ A2

$+\int_{12}^{\frac{27}{2}}\left(e^{\frac{1}{2}\sqrt{x}} - e^{\frac{1}{2}\sqrt{x}} \sin\left(\frac{\pi}{3}x\right)\right)\mathrm{d}x$

[4]

11. (a) $\vec{PS} = \vec{PQ} + \vec{QS}$

$\vec{PS} = \mathbf{r} + \frac{1}{\alpha+1}\vec{QR}$ (A1) for correct approach

$\vec{PS} = \mathbf{r} + \frac{1}{\alpha+1}(\vec{PR} - \vec{PQ})$ (M1) for valid approach

$\vec{PS} = \mathbf{r} + \frac{1}{\alpha+1}(\mathbf{q} - \mathbf{r})$

$\vec{PS} = \frac{\alpha+1}{\alpha+1}\mathbf{r} + \frac{1}{\alpha+1}\mathbf{q} - \frac{1}{\alpha+1}\mathbf{r}$ (M1) for valid approach

$\vec{PS} = \frac{1}{\alpha+1}\mathbf{q} + \frac{\alpha}{\alpha+1}\mathbf{r}$ A1

[4]

(b) $\because \mathrm{PS} \perp \mathrm{QR}$

$\therefore \vec{PS} \cdot \vec{QR} = 0$ M1

$\left(\frac{1}{\alpha+1}\mathbf{q} + \frac{\alpha}{\alpha+1}\mathbf{r}\right) \cdot (\mathbf{q} - \mathbf{r}) = 0$ A1

$(\mathbf{q} + \alpha\mathbf{r}) \cdot (\mathbf{q} - \mathbf{r}) = 0$

$\mathbf{q} \cdot (\mathbf{q} - \mathbf{r}) + \alpha\mathbf{r} \cdot (\mathbf{q} - \mathbf{r}) = 0$ M1

$\alpha\mathbf{r} \cdot (\mathbf{q} - \mathbf{r}) = -\mathbf{q} \cdot (\mathbf{q} - \mathbf{r})$

$\alpha\mathbf{r} \cdot (\mathbf{q} - \mathbf{r}) = \mathbf{q} \cdot (\mathbf{r} - \mathbf{q})$ M1

$\alpha = \frac{\mathbf{q} \cdot (\mathbf{r} - \mathbf{q})}{\mathbf{r} \cdot (\mathbf{q} - \mathbf{r})}$ AG

[4]

(c) $\vec{PS} = \frac{1}{\alpha+1}\mathbf{q} + \frac{\alpha}{\alpha+1}\mathbf{r}$

$\vec{PS} = \frac{1}{\alpha+1}(\mathbf{q} + \alpha\mathbf{r})$

$\vec{PS} = \frac{1}{\frac{\mathbf{q} \cdot (\mathbf{r} - \mathbf{q})}{\mathbf{r} \cdot (\mathbf{q} - \mathbf{r})} + 1}\left(\mathbf{q} + \frac{\mathbf{q} \cdot (\mathbf{r} - \mathbf{q})}{\mathbf{r} \cdot (\mathbf{q} - \mathbf{r})}\mathbf{r}\right)$ M1

$\vec{PS} = \frac{1}{\frac{\mathbf{q} \cdot (\mathbf{r} - \mathbf{q})}{\mathbf{r} \cdot (\mathbf{q} - \mathbf{r})} + \frac{\mathbf{r} \cdot (\mathbf{q} - \mathbf{r})}{\mathbf{r} \cdot (\mathbf{q} - \mathbf{r})}}\left(\frac{\mathbf{r} \cdot (\mathbf{q} - \mathbf{r})}{\mathbf{r} \cdot (\mathbf{q} - \mathbf{r})}\mathbf{q} + \frac{\mathbf{q} \cdot (\mathbf{r} - \mathbf{q})}{\mathbf{r} \cdot (\mathbf{q} - \mathbf{r})}\mathbf{r}\right)$ M1

$\vec{PS} = \frac{\mathbf{r} \cdot (\mathbf{q} - \mathbf{r})}{\mathbf{q} \cdot (\mathbf{r} - \mathbf{q}) + \mathbf{r} \cdot (\mathbf{q} - \mathbf{r})}\left(\frac{\mathbf{r} \cdot (\mathbf{q} - \mathbf{r})}{\mathbf{r} \cdot (\mathbf{q} - \mathbf{r})}\mathbf{q} + \frac{\mathbf{q} \cdot (\mathbf{r} - \mathbf{q})}{\mathbf{r} \cdot (\mathbf{q} - \mathbf{r})}\mathbf{r}\right)$ A1

$$\vec{PS}=\frac{(\mathbf{r}\cdot(\mathbf{q}-\mathbf{r}))\mathbf{q}+(\mathbf{q}\cdot(\mathbf{r}-\mathbf{q}))\mathbf{r}}{\mathbf{q}\cdot(\mathbf{r}-\mathbf{q})+\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})}$$ M1

$$\vec{PS}=\frac{(\mathbf{r}\cdot(\mathbf{q}-\mathbf{r}))\mathbf{q}+(\mathbf{q}\cdot(\mathbf{r}-\mathbf{q}))\mathbf{r}}{\mathbf{q}\cdot(\mathbf{r}-\mathbf{q})-\mathbf{r}\cdot(\mathbf{r}-\mathbf{q})}$$ A1

$$\vec{PS}=\frac{(\mathbf{r}\cdot(\mathbf{q}-\mathbf{r}))\mathbf{q}+(\mathbf{q}\cdot(\mathbf{r}-\mathbf{q}))\mathbf{r}}{(\mathbf{q}-\mathbf{r})\cdot(\mathbf{r}-\mathbf{q})}$$

$$\vec{PS}=-\frac{(\mathbf{r}\cdot(\mathbf{q}-\mathbf{r}))\mathbf{q}+(\mathbf{q}\cdot(\mathbf{r}-\mathbf{q}))\mathbf{r}}{|\mathbf{r}-\mathbf{q}|^2}$$ A1

$$\vec{PS}=\frac{(\mathbf{r}\cdot(\mathbf{r}-\mathbf{q}))\mathbf{q}-(\mathbf{q}\cdot(\mathbf{r}-\mathbf{q}))\mathbf{r}}{|\mathbf{r}-\mathbf{q}|^2}$$ AG

[6]

(d) (i)

$$\alpha=\frac{\mathbf{q}\cdot(\mathbf{r}-\mathbf{q})}{\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})}$$

$$\alpha=\frac{\mathbf{q}\cdot\mathbf{r}-\mathbf{q}\cdot\mathbf{q}}{\mathbf{r}\cdot\mathbf{q}-\mathbf{r}\cdot\mathbf{r}}$$ (M1) for valid approach

$$\alpha=\frac{\mathbf{q}\cdot\mathbf{r}-|\mathbf{q}|^2}{\mathbf{r}\cdot\mathbf{q}-|\mathbf{r}|^2}$$

$$\alpha=\frac{0-20^2}{0-15^2}$$ (A1) for substitution

$$\alpha=\frac{16}{9}$$ A1

(ii)

$$QR=\sqrt{20^2+15^2}$$

$QR=25$ (A1) for correct value

$$RS=25\left(\frac{\frac{16}{9}}{\frac{16}{9}+1}\right)$$

$RS=16$ (A1) for correct value

$$PS=\sqrt{20^2-16^2}$$

$PS=12$ (A1) for correct value

The required area

$$=\frac{(16)(12)}{2}$$

$=96$ A1

[7]

12. (a)

$R^2 = \mathrm{OP}^2 + r^2$ — (M1) for valid approach

$R^2 = (h-R)^2 + r^2$

$R^2 = h^2 - 2Rh + R^2 + r^2$

$2Rh - h^2 = r^2$ — A1

$V = \frac{1}{3}\pi r^2 h$

$\therefore V = \frac{1}{3}\pi(2Rh - h^2)(h)$ — (A1) for substitution

$\therefore V = \frac{2R}{3}\pi h^2 - \frac{1}{3}\pi h^3$ — A1

[4]

(b)

$V = \frac{2R}{3}\pi h^2 - \frac{1}{3}\pi h^3$

$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{2R}{3}\pi(2h) - \frac{1}{3}\pi(3h^2)$ — M1A1

$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{4R}{3}\pi h - \pi h^2$ — A1

$\frac{\mathrm{d}^2V}{\mathrm{d}h^2} = \frac{4R}{3}\pi(1) - \pi(2h)$ — A1

$\frac{\mathrm{d}^2V}{\mathrm{d}h^2} = \frac{4R}{3}\pi - 2\pi h$ — AG

[4]

(c)

$\frac{\mathrm{d}V}{\mathrm{d}h} = 0$

$\therefore \frac{4R}{3}\pi h - \pi h^2 = 0$ — M1

$\frac{4R}{3} - h = 0$

$h = \frac{4R}{3}$ — A1

$\left.\frac{\mathrm{d}^2V}{\mathrm{d}h^2}\right|_{h=\frac{4R}{3}} = \frac{4R}{3}\pi - 2\pi\left(\frac{4R}{3}\right)$ — M1

$\left.\frac{\mathrm{d}^2V}{\mathrm{d}h^2}\right|_{h=\frac{4R}{3}} = -\frac{4R}{3}\pi$

$\left.\frac{\mathrm{d}^2V}{\mathrm{d}h^2}\right|_{h=\frac{4R}{3}} < 0$ — R1

Thus, V attains its maximum when $h = \frac{4R}{3}$.

$2R\left(\frac{4R}{3}\right)-\left(\frac{4R}{3}\right)^2=r^2$ A1

$\frac{8R^2}{3}-\frac{16R^2}{9}=r^2$ M1

$\frac{8R^2}{9}=r^2$

$r=\frac{2\sqrt{2}R}{3}$

Thus, V attains its maximum when $r=\frac{2\sqrt{2}R}{3}$. AG

[6]

(d) $\frac{32}{81}\pi R^3$ A2

[2]

(e) The slant height of the circular cone

$=\sqrt{\left(\frac{2\sqrt{2}R}{3}\right)^2+\left(\frac{4R}{3}\right)^2}$ (M1) for valid approach

$=\sqrt{\frac{24}{9}R^2}$

$=\frac{\sqrt{24}R}{3}$

$=\frac{2\sqrt{6}R}{3}$ A1

The curved surface area of the circular cone

$=\pi\left(\frac{2\sqrt{2}R}{3}\right)\left(\frac{2\sqrt{6}R}{3}\right)$

$=\frac{4}{9}\sqrt{12}\pi R^2$

$<\frac{4}{9}(4)\pi R^2$ R1

$=\frac{16}{9}\pi R^2$

Thus, the curved surface area of the circular cone is not greater than $\frac{16}{9}\pi R^2$ when its volume attains its maximum. A1

[4]