Chapter 15 Solution

Exercise 66

1. (a) $\frac{dx}{dt} = \pi^2 x \sin \pi t$

$\frac{1}{x} dx = \pi^2 \sin \pi t dt$ (M1) for valid approach

$\int \frac{1}{x} dx = \int \pi^2 \sin \pi t dt$ (A1) for correct approach

Let $u = \pi t$. (M1) for substitution

$\frac{du}{dt} = \pi \Rightarrow du = \pi dt$

$\therefore \int \frac{1}{x} dx = \int \pi \sin u du$ (A1) for correct working

$\ln|x| = -\pi \cos u + C$

$\ln|x| = -\pi \cos \pi t + C$

$x = e^{-\pi \cos \pi t + C}$ A1

$1 = e^{-\pi \cos \pi (0.5) + C}$ (M1) for substitution

$1 = e^C$

$C = 0$ (A1) for correct value

$\therefore x = e^{-\pi \cos \pi t}$ A1

[8]

(b) $e^{-\pi} \le x \le e^{\pi}$ A2

[2]

2. $\dfrac{dy}{dx} = \dfrac{B}{y^2}$

$y^2 dy = B dx$	(M1) for valid approach
$\int y^2 dy = \int B dx$	(A1) for correct approach
$\dfrac{1}{3} y^3 = Bx + C$	A1
$y^3 = 3Bx + C$	
$y = (3Bx + C)^{\frac{1}{3}}$	A1
$3 = (3B(4) + C)^{\frac{1}{3}}$	(M1) for substitution
$27 = 12B + C$	
$C = 27 - 12B$	
$9 = (3B(121) + C)^{\frac{1}{3}}$	
$\therefore 729 = 363B + 27 - 12B$	(M1) for substitution
$702 = 351B$	
$B = 2$	
$C = 27 - 12(2)$	
$C = 3$	(A1) for correct value
$\therefore y = \left(3(2)\left(\dfrac{61}{3}\right) + 3\right)^{\frac{1}{3}}$	
$y = 125^{\frac{1}{3}}$	
$y = 5$	A1

[8]

3. $\frac{dy}{dx} = -y^3 \sin^2 x$

$-\frac{1}{y^3}dy = \sin^2 x dx$ (M1) for valid approach

$\int -\frac{1}{y^3}dy = \int \sin^2 x dx$ (A1) for correct approach

$\int -\frac{1}{y^3}dy = \int \frac{1-\cos 2x}{2}dx$ (A1) for correct approach

$\int -\frac{1}{y^3}dy = \frac{1}{2}\int (1-\cos 2x)dx$

$\frac{1}{2y^2} = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$ A1

$\frac{1}{y^2} = x - \frac{1}{2}\sin 2x + C$

$y = \frac{1}{\sqrt{x - \frac{1}{2}\sin 2x + C}}$ A1

$2 = \frac{1}{\sqrt{0 - \frac{1}{2}\sin 2(0) + C}}$ (M1) for substitution

$2 = \frac{1}{\sqrt{C}}$

$\sqrt{C} = \frac{1}{2}$

$C = \frac{1}{4}$ (A1) for correct value

$\therefore y = \frac{1}{\sqrt{x - \frac{1}{2}\sin 2x + \frac{1}{4}}}$

$y = \frac{1}{\sqrt{\frac{1}{4}(4x - 2\sin 2x + 1)}}$

$y = \frac{2}{\sqrt{4x - 2\sin 2x + 1}}$ A1

[8]

4. $\frac{dy}{dx} = \frac{y}{(2x+3)\ln y}$

$\frac{\ln y}{y}dy = \frac{1}{2x+3}dx$ (M1) for valid approach

$\int \frac{\ln y}{y}dy = \int \frac{1}{2x+3}dx$ (A1) for correct approach

Let $u = \ln y$. (M1) for substitution

$\frac{du}{dy} = \frac{1}{y} \Rightarrow du = \frac{1}{y}dy$

$\therefore \int u du = \int \frac{1}{2x+3}dx$ (A1) for correct working

$\frac{1}{2}u^2 = \frac{1}{2}\ln|2x+3| + C$

$(\ln y)^2 = \ln|2x+3| + C$ (M1) for valid approach

$\ln y = \sqrt{\ln|2x+3| + C}$

$y = e^{\sqrt{\ln|2x+3|+C}}$ A1

$e^{\sqrt{\ln 5}} = e^{\sqrt{\ln|2(1)+3|+C}}$ (M1) for substitution

$e^{\sqrt{\ln 5}} = e^{\sqrt{\ln 5+C}}$

$\sqrt{\ln 5} = \sqrt{\ln 5 + C}$

$C = 0$ (A1) for correct value

$\therefore y = e^{\sqrt{\ln|2x+3|}}$ A1

[9]

Exercise 67

1. $\frac{dy}{dx} - \frac{y}{x} = \frac{1}{x^6}$

$\frac{dy}{dx} - \frac{1}{x}y = \frac{1}{x^6}$

The integrating factor

$= e^{\int -\frac{1}{x}dx}$ (M1) for valid approach

$= e^{-\ln x}$

$= e^{\ln x^{-1}}$

$= \frac{1}{x}$ A1

$\therefore \frac{1}{x}\frac{dy}{dx} - \frac{1}{x}\cdot\frac{1}{x}y = \frac{1}{x}\cdot\frac{1}{x^6}$ (M1) for valid approach

$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x^7}$

$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x^7}$ (A1) for correct approach

$\frac{y}{x} = \int \frac{1}{x^7} dx$

$\frac{y}{x} = -\frac{1}{6x^6} + C$

$y = -\frac{1}{6x^5} + Cx$ A1

$\frac{11}{6} = -\frac{1}{6(1)^5} + C(1)$ (M1) for substitution

$C = 2$ (A1) for correct value

$\therefore y = -\frac{1}{6x^5} + 2x$ A1

[8]

2. $x\frac{dy}{dx} - y = x^2 2^x$

$\frac{dy}{dx} - \frac{1}{x}y = x2^x$ (A1) for correct approach

The integrating factor

$= e^{\int -\frac{1}{x}dx}$ (M1) for valid approach

$= e^{-\ln x}$

$= e^{\ln x^{-1}}$

$= \frac{1}{x}$ A1

$\therefore \frac{1}{x}\frac{dy}{dx} - \frac{1}{x}\cdot\frac{1}{x}y = \frac{1}{x}\cdot x2^x$ (M1) for valid approach

$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 2^x$

$\frac{d}{dx}\left(\frac{y}{x}\right) = 2^x$ (A1) for correct approach

$\frac{y}{x} = \int 2^x dx$

$\frac{y}{x} = \frac{1}{\ln 2}2^x + C$

$y = \frac{1}{\ln 2}x2^x + Cx$ A1

$\frac{3}{\ln 2} = \frac{1}{\ln 2}(1)2^1 + C(1)$ (M1) for substitution

$C = \frac{1}{\ln 2}$ (A1) for correct value

$\therefore y = \frac{1}{\ln 2}x2^x + \frac{1}{\ln 2}x$

$y = \frac{1}{\ln 2}x(2^x + 1)$ A1

[9]

3. $\frac{dy}{dx} = \cos^3 x - 2y\tan x$

$\frac{dy}{dx} + (2\tan x)y = \cos^3 x$

Let $u = \cos x$. (M1) for substitution

$\frac{du}{dx} = -\sin x \Rightarrow (-1)\cdot du = \sin x dx$

The integrating factor

$= e^{\int 2\tan x dx}$ (M1) for valid approach

$= e^{\int \frac{2\sin x}{\cos x} dx}$

$= e^{\int -\frac{2}{u} du}$

$= e^{-2\ln u}$

$= e^{\ln u^{-2}}$

$= \frac{1}{u^2}$

$= \frac{1}{\cos^2 x}$ A1

$\therefore \frac{1}{\cos^2 x}\frac{dy}{dx} + \frac{1}{\cos^2 x}(2\tan x)y = \frac{1}{\cos^2 x}\cdot\cos^3 x$ (M1) for valid approach

$\frac{1}{\cos^2 x}\frac{dy}{dx} + \frac{2\sin x}{\cos^3 x}\cdot y = \cos x$

$\frac{d}{dx}\left(\frac{y}{\cos^2 x}\right) = \cos x$ (A1) for correct approach

$\frac{y}{\cos^2 x} = \int \cos x dx$

$\frac{y}{\cos^2 x} = \sin x + C$

$y = \cos^2 x(\sin x + C)$ A1

$3 = \cos^2 0(\sin 0 + C)$ (M1) for substitution

$3 = 0 + C$

$C = 3$ (A1) for correct value

$\therefore y = \cos^2 x(\sin x + 3)$ A1

[9]

4. $$\sin x\frac{dy}{dx}+\cos^2 x=1-y\cos x$$

$$\sin x\frac{dy}{dx}+y\cos x=1-\cos^2 x$$

$$\sin x\frac{dy}{dx}+y\cos x=\sin^2 x$$ A1

$$\frac{dy}{dx}+(\cot x)y=\sin x$$ A1

Let $u=\sin x$. M1

$$\frac{du}{dx}=\cos x\Rightarrow du=\cos x dx$$

The integrating factor

$$=e^{\int \cot x dx}$$ M1

$$=e^{\int \frac{\cos x}{\sin x}dx}$$

$$=e^{\int \frac{1}{u}du}$$

$$=e^{\ln u}$$

$$=u$$

$$=\sin x$$ A1

$$\therefore \sin x\frac{dy}{dx}+\sin x\cdot(\cot x)y=\sin x\cdot\sin x$$ M1

$$\sin x\frac{dy}{dx}+(\cos x)y=\sin^2 x$$

$$\frac{d}{dx}(y\sin x)=\sin^2 x$$ A1

$$y\sin x=\int \sin^2 x dx$$

$$y\sin x=\int \frac{1}{2}(1-\cos 2x)dx$$ A1

$$y\sin x=\int\left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)dx$$

$$y\sin x=\frac{1}{2}x-\frac{1}{4}\sin 2x+C$$

$$y\sin x=\frac{2x-\sin 2x+C}{4}$$

$$y=\frac{2x-\sin 2x+C}{4\sin x}$$ A1

$$\pi=\frac{2\left(\frac{\pi}{2}\right)-\sin 2\left(\frac{\pi}{2}\right)+C}{4\sin\frac{\pi}{2}}$$ M1

$$\pi = \frac{\pi + C}{4}$$

$$4\pi = \pi + C$$

$$C = 3\pi$$

$$\therefore y = \frac{2x - \sin 2x + 3\pi}{4\sin x}$$ AG

[10]

Exercise 68

1. $\begin{cases} x_{n+1} = x_n + 0.1 \\ y_{n+1} = y_n + 0.1 \left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x_n,\, y_n)} \end{cases}$ (M1) for valid approach

$x_0 = 4,\ y_0 = 10$ (A1) for correct values

$x_1 = 4 + 0.1 = 4.1$

$y_1 = 10 + 0.1\left(\dfrac{(4)(10)}{4^2 - 9}\right) = 10.57142857$ A1

$x_2 = 4.1 + 0.1 = 4.2$

$y_2 = 10.57142857 + 0.1\left(\dfrac{(4.1)(10.57142857)}{4.1^2 - 9}\right)$

$y_2 = 11.12639473$ A1

$x_3 = 4.2 + 0.1 = 4.3$

$y_3 = 11.12639473 + 0.1\left(\dfrac{(4.2)(11.12639473)}{4.2^2 - 9}\right)$

$y_3 = 11.66726114$ A1

Thus, the required approximation is 11.7. A1

[6]

2. $\begin{cases} x_{n+1} = x_n + 0.2 \\ y_{n+1} = y_n + 0.2 \left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x_n,\, y_n)} \end{cases}$ (M1) for valid approach

$x_0 = 0,\ y_0 = 2$ (A1) for correct values

$x_1 = 0 + 0.2 = 0.2$

$y_1 = 2 + 0.2(e^0 + 4(2)) = 3.8$ A1

$x_2 = 0.2 + 0.2 = 0.4$

$y_2 = 3.8 + 0.2(e^{0.2} + 4(3.8)) = 7.084280552$ A1

$x_3 = 0.4 + 0.2 = 0.6$

$y_3 = 7.084280552 + 0.2(e^{0.4} + 4(7.084280552))$

$y_3 = 13.05006993$ A1

$x_4 = 0.6 + 0.2 = 0.8$

$y_4 = 13.05006993 + 0.2(e^{0.6} + 4(13.05006993))$

$y_4 = 23.85454964$ A1

Thus, the required approximation is 23.9. A1

[7]

3. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \frac{1}{x^3}$

$$\begin{cases} x_{n+1} = x_n + 0.25 \\ y_{n+1} = y_n + 0.25 \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x_n,\, y_n)} \end{cases}$$ (M1) for valid approach

$x_0 = 1,\ y_0 = 2$ (A1) for correct values

$x_1 = 1 + 0.25 = 1.25$

$y_1 = 2 + 0.25\left(\frac{2}{1} + \frac{1}{1^3}\right) = 2.75$ A1

$x_2 = 1.25 + 0.25 = 1.5$

$y_2 = 2.75 + 0.25\left(\frac{2.75}{1.25} + \frac{1}{1.25^3}\right) = 3.428$ A1

$x_3 = 1.5 + 0.25 = 1.75$

$y_3 = 3.428 + 0.25\left(\frac{3.428}{1.5} + \frac{1}{1.5^3}\right) = 4.073407407$ A1

$x_4 = 1.75 + 0.25 = 2$

$y_4 = 4.073407407 + 0.25\left(\frac{4.073407407}{1.75} + \frac{1}{1.75^3}\right)$

$y_4 = 4.701969982$ A1

Thus, the required approximation is 4.70. A1

[7]

4. $\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - y = -1$

$\frac{\mathrm{d}y}{\mathrm{d}x} - xy = -x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = xy - x$ (M1) for valid approach

$\begin{cases} x_{n+1} = x_n + 0.2 \\ y_{n+1} = y_n + 0.2\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x_n,\, y_n)} \end{cases}$ (M1) for valid approach

$x_0 = 3,\ y_0 = 3$ (A1) for correct values

$x_1 = 3 + 0.2 = 3.2$

$y_1 = 3 + 0.2((3)(3) - 3) = 4.2$ A1

$x_2 = 3.2 + 0.2 = 3.4$

$y_2 = 4.2 + 0.2((3.2)(4.2) - 3.2) = 6.248$ A1

$x_3 = 3.4 + 0.2 = 3.6$

$y_3 = 6.248 + 0.2((3.4)(6.248) - 3.4) = 9.81664$ A1

$x_4 = 3.6 + 0.2 = 3.8$

$y_4 = 9.81664 + 0.2((3.6)(9.81664) - 3.6) = 16.1646208$ A1

$x_5 = 3.8 + 0.2 = 4$

$y_5 = 16.1646208 + 0.2((3.8)(16.1646208) - 3.8)$

$y_5 = 27.68973261$ A1

Thus, the required approximation is 27.7. A1

[9]

Exercise 69

1. $\frac{dy}{dx} - 4y = e^{2x}$

$\frac{dy}{dx} = 4y + e^{2x}$

$\left.\frac{dy}{dx}\right|_{x=0} = 4(1) + e^{2(0)} = 5$ (A1) for correct value

$\frac{d^2y}{dx^2} = 4\frac{dy}{dx} + 2e^{2x}$ (A1) for correct approach

$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 4(5) + 2e^{2(0)} = 22$ (A1) for correct value

$y = y(0) + x\left.\frac{dy}{dx}\right|_{x=0} + \frac{x^2}{2!}\left.\frac{d^2y}{dx^2}\right|_{x=0} + \cdots$

$y = 1 + x(5) + \frac{x^2}{2}(22) + \cdots$ M1

$y = 1 + 5x + 11x^2 + \cdots$ A1

[5]

2. $\frac{dy}{dx} - xy = \ln(x+1)$

$\frac{dy}{dx} = xy + \ln(x+1)$

$\left.\frac{dy}{dx}\right|_{x=0} = (0)(-2) + \ln(0+1) = 0$ (A1) for correct value

$\frac{d^2y}{dx^2} = (1)(y) + x\frac{dy}{dx} + \frac{1}{x+1} = y + x\frac{dy}{dx} + \frac{1}{x+1}$ (A1) for correct approach

$\left.\frac{d^2y}{dx^2}\right|_{x=0} = -2 + (0)(0) + \frac{1}{0+1} = -1$ (A1) for correct value

$y = y(0) + x\left.\frac{dy}{dx}\right|_{x=0} + \frac{x^2}{2!}\left.\frac{d^2y}{dx^2}\right|_{x=0} + \cdots$

$y = -2 + x(0) + \frac{x^2}{2}(-1) + \cdots$ M1

$y = -2 - \frac{1}{2}x^2 + \cdots$ A1

[5]

3. $e^{2x}\frac{dy}{dx}-e^{x}y=1$

$\frac{dy}{dx}-e^{-x}y=e^{-2x}$

$\frac{dy}{dx}=e^{-x}y+e^{-2x}$ (M1) for valid approach

$\left.\frac{dy}{dx}\right|_{x=0}=e^{-0}(e)+e^{-2(0)}=e+1$ (A1) for correct value

$\frac{d^2y}{dx^2}=(-e^{-x})(y)+e^{-x}\frac{dy}{dx}-2e^{2x}=-e^{-x}y+e^{-x}\frac{dy}{dx}-2e^{2x}$ (A1) for correct approach

$\left.\frac{d^2y}{dx^2}\right|_{x=0}=-e^{-0}(e)+e^{-0}(e+1)-2e^{2(0)}=-1$ (A1) for correct value

$y=y(0)+x\left.\frac{dy}{dx}\right|_{x=0}+\frac{x^2}{2!}\left.\frac{d^2y}{dx^2}\right|_{x=0}+\cdots$

$y=e+x(e+1)+\frac{x^2}{2}(-1)+\cdots$ M1

$y=e+(e+1)x-\frac{1}{2}x^2+\cdots$ A1

[6]

4. $e^{3x}\dfrac{dy}{dx}-(3x^2+2x-1)e^{3x}y-e^{-3x}=0$

$\dfrac{dy}{dx}-(3x^2+2x-1)y-e^{-6x}=0$

$\dfrac{dy}{dx}=(3x^2+2x-1)y+e^{-6x}$ (M1) for valid approach

$\left.\dfrac{dy}{dx}\right|_{x=0}=(3(0)^2+2(0)-1)(2)+e^{-6(0)}=-1$ (A1) for correct value

$\dfrac{d^2y}{dx^2}=(6x+2)(y)+(3x^2+2x-1)\dfrac{dy}{dx}-6e^{-6x}$

$\dfrac{d^2y}{dx^2}=(6x+2)y+(3x^2+2x-1)\dfrac{dy}{dx}-6e^{-6x}$ (A1) for correct approach

$\left.\dfrac{d^2y}{dx^2}\right|_{x=0}=(6(0)+2)(2)+(3(0)^2+2(0)-1)(-1)-6e^{-6(0)}=-1$ (A1) for correct value

$\dfrac{d^3y}{dx^3}=(6)(y)+(6x+2)\dfrac{dy}{dx}$

$+(6x+2)\dfrac{dy}{dx}+(3x^2+2x-1)\dfrac{d^2y}{dx^2}-6(-6)e^{-6x}$

$\dfrac{d^3y}{dx^3}=6y+(12x+4)\dfrac{dy}{dx}+(3x^2+2x-1)\dfrac{d^2y}{dx^2}+36e^{-6x}$ (A1) for correct approach

$\left.\dfrac{d^3y}{dx^3}\right|_{x=0}=6(2)+(12(0)+4)(-1)$

$+(3(0)^2+2(0)-1)(-1)+36e^{-6(0)}$

$\left.\dfrac{d^3y}{dx^3}\right|_{x=0}=45$ (A1) for correct value

$y=y(0)+x\left.\dfrac{dy}{dx}\right|_{x=0}+\dfrac{x^2}{2!}\left.\dfrac{d^2y}{dx^2}\right|_{x=0}+\dfrac{x^3}{3!}\left.\dfrac{d^3y}{dx^3}\right|_{x=0}+\cdots$

$y=2+x(-1)+\dfrac{x^2}{2}(-1)+\dfrac{x^3}{6}(45)+\cdots$ M1

$y=2-x-\dfrac{1}{2}x^2+\dfrac{15}{2}x^3+\cdots$ A1

[8]

Exercise 70

1. (a) $\frac{\mathrm{d}v}{\mathrm{d}t}=\sqrt{4-v^2}$

$\frac{1}{\sqrt{4-v^2}}\mathrm{d}v=\mathrm{d}t$ (M1) for valid approach

$\int\frac{1}{\sqrt{4-v^2}}\mathrm{d}v=\int\mathrm{d}t$ (A1) for correct approach

$\int\frac{1}{\sqrt{2^2-v^2}}\mathrm{d}v=\int\mathrm{d}t$

$\arcsin\frac{v}{2}=t+C$ A1

$\frac{v}{2}=\sin(t+C)$

$v=2\sin(t+C)$ A1

$2=2\sin(0+C)$ (M1) for substitution

$1=\sin C$

$\sin C=\sin\frac{\pi}{2}$

$C=\frac{\pi}{2}$ (A1) for correct value

$\therefore v=2\sin\left(t+\frac{\pi}{2}\right)$ A1

[7]

(b) The total distance travelled

$=\int_0^{\frac{2\pi}{3}}|v(t)|\,\mathrm{d}t$ (M1) for valid approach

$=\int_0^{\frac{2\pi}{3}}\left|2\sin\left(t+\frac{\pi}{2}\right)\right|\mathrm{d}t$ (A1) for substitution

$=2.267949192\text{ m}$

$=2.27\text{ m}$ A1

[3]

(c) $s = \int v \mathrm{d}t$

$s = \int 2\sin\left(t + \frac{\pi}{2}\right)\mathrm{d}t$ (M1) for valid approach

$s = -2\cos\left(t + \frac{\pi}{2}\right) + D$ A1

$0 = -2\cos\left(\frac{\pi}{2} + \frac{\pi}{2}\right) + D$ (M1) for substitution

$0 = 2 + D$

$D = -2$ (A1) for correct value

$\therefore s = -2\cos\left(t + \frac{\pi}{2}\right) - 2$ A1

[5]

(d) $s = -2\cos\left(t + \frac{\pi}{2}\right) - 2$

$s + 2 = -2\cos\left(t + \frac{\pi}{2}\right)$

$-\frac{s+2}{2} = \cos\left(t + \frac{\pi}{2}\right)$ A1

$v = 2\sin\left(t + \frac{\pi}{2}\right)$

$v = 2\sqrt{1 - \cos^2\left(t + \frac{\pi}{2}\right)}$ M1

$\therefore v = 2\sqrt{1 - \left(-\frac{s+2}{2}\right)^2}$ A1

$v = 2\sqrt{1 - \frac{s^2 + 4s + 4}{4}}$

$v = 2\sqrt{\frac{4 - s^2 - 4s - 4}{4}}$ M1

$v = \frac{2\sqrt{-s^2 - 4s}}{2}$ A1

$v = \sqrt{-s(s+4)}$ AG

[5]

2. (a)

$$a = \frac{v+150}{300}$$

$$\frac{dv}{dt} = \frac{v+150}{300}$$

$$\frac{1}{v+150}\,dv = \frac{1}{300}\,dt$$ (M1) for valid approach

$$\int \frac{1}{v+150}\,dv = \int \frac{1}{300}\,dt$$ (A1) for correct approach

$$\ln|v+150| = \frac{1}{300}t + C$$ A1

$$v+150 = e^{\frac{1}{300}t+C}$$

$$v = e^{\frac{1}{300}t+C} - 150$$

$$v = e^{C} \cdot e^{\frac{1}{300}t} - 150$$ A1

$$0 = e^{C} \cdot e^{\frac{1}{300}(0)} - 150$$ (M1) for substitution

$$e^{C} = 150$$ (A1) for correct value

$$\therefore v = 150e^{\frac{1}{300}t} - 150$$ A1

[7]

(b)

$$\ln(v+150) = \frac{1}{300}t + \ln 150$$

$$\ln(5+150) = \frac{1}{300}t + \ln 150$$ (M1) for setting equation

$$\ln 155 - \ln 150 = \frac{1}{300}t$$

$$\ln\frac{31}{30} = \frac{1}{300}t$$ (A1) for correct approach

$$t = 300\ln\frac{31}{30}\text{ s}$$ A1

[3]

(c) $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{v+150}{300}$

$\frac{\mathrm{d}v}{\mathrm{d}s} \cdot \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{v+150}{300}$ A1

$v\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{v+150}{300}$ A1

$\frac{300v}{v+150}\mathrm{d}v = \mathrm{d}s$ M1

$\int \frac{300v}{v+150}\mathrm{d}v = \int \mathrm{d}s$ A1

$\int \frac{300(v+150-150)}{v+150}\mathrm{d}v = \int \mathrm{d}s$ M1

$\int \left(300 - \frac{150}{v+150}\right)\mathrm{d}v = \int \mathrm{d}s$

$s = \int \left(300 - \frac{150}{v+150}\right)\mathrm{d}v$ AG

[5]

(d) $s = \int \left(300 - \frac{150}{v+150}\right)\mathrm{d}v$

$s = 300v - 150\ln|v+150| + D$ A1

$0 = 300(0) - 150\ln(0+150) + D$ (M1) for substitution

$D = 150\ln 150$ (A1) for correct value

$\therefore s = 300v - 150\ln(v+150) + 150\ln 150$

$s = 300(5) - 150\ln(5+150) + 150\ln 150$ (M1) for substitution

$s = 1500 - 150(\ln 155 - \ln 150)$

$s = 150\left(10 - \ln\frac{31}{30}\right)$ m A1

[5]

3. (a) (i) Let $\frac{1}{x(x+3)} \equiv \frac{A}{x} + \frac{B}{x+3}$, where A and B are constants.

$\frac{1}{x(x+3)} \equiv \frac{A(x+3)}{x(x+3)} + \frac{Bx}{x(x+3)}$	M1
$\frac{1}{x(x+3)} \equiv \frac{Ax+3A+Bx}{x(x+3)}$	
$1 \equiv (A+B)x + 3A$	A1
$1 = 3A$	
$A = \frac{1}{3}$	A1
$0 = \frac{1}{3} + B$	
$B = -\frac{1}{3}$	
$\therefore \frac{1}{x(x+3)} \equiv \frac{1}{3x} - \frac{1}{3(x+3)}$	A1

(ii)

$\frac{dv}{dt} = -\frac{v(v+3)}{3}$	
$\frac{1}{v(v+3)} dv = -\frac{1}{3} dt$	(M1) for valid approach
$\int \frac{1}{v(v+3)} dv = \int -\frac{1}{3} dt$	(A1) for correct approach
$\therefore \frac{1}{3} \int \left(\frac{1}{v} - \frac{1}{v+3} \right) dv = \int -\frac{1}{3} dt$	(M1) for substitution
$\int \left(\frac{1}{v} - \frac{1}{v+3} \right) dv = \int -dt$	
$\ln\lvert v\rvert - \ln\lvert v+3\rvert = -t + C$	A1
$\ln\left\lvert \frac{v}{v+3} \right\rvert = -t + C$	
$\frac{v}{v+3} = e^{-t+C}$	(A1) for correct approach
$v = (v+3)e^{-t+C}$	
$v = ve^{-t+C} + 3e^{-t+C}$	
$v - ve^{-t+C} = 3e^{-t+C}$	(M1) for valid approach
$v(1-e^{-t+C}) = 3e^{-t+C}$	
$v = \frac{3e^{-t+C}}{1-e^{-t+C}}$	A1

$$1.5=\frac{3e^{0+C}}{1-e^{0+C}}$$ (M1) for substitution

$$1.5(1-e^{C})=3e^{C}$$

$$1.5-1.5e^{C}=3e^{C}$$

$$1.5=4.5e^{C}$$

$$e^{C}=\frac{1}{3}$$ (A1) for correct value

$$\therefore v=\frac{3e^{-t}\left(\frac{1}{3}\right)}{1-e^{-t}\left(\frac{1}{3}\right)}$$

$$v=\frac{3e^{-t}}{3-e^{-t}}$$ A1

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(b) $\frac{\mathrm{d}v}{\mathrm{d}t}=-\frac{v(v+3)}{3}$

$$\frac{\mathrm{d}v}{\mathrm{d}s}\cdot\frac{\mathrm{d}s}{\mathrm{d}t}=-\frac{v(v+3)}{3}$$ A1

$$v\frac{\mathrm{d}v}{\mathrm{d}s}=-\frac{v(v+3)}{3}$$ A1

$$-\frac{3}{v+3}\mathrm{d}v=\mathrm{d}s$$ M1

$$\int-\frac{3}{v+3}\mathrm{d}v=\int\mathrm{d}s$$ A1

$$s=-3\ln|v+3|+D$$ A1

$$-2\ln 4.5=-3\ln(1.5+3)+D$$ M1

$$-2\ln 4.5=-3\ln 4.5+D$$

$$D=\ln 4.5$$ A1

$$\therefore s=-3\ln(v+3)+\ln 4.5$$

$$s=\ln\frac{1}{(v+3)^{3}}+\ln 4.5$$ A1

$$s=\ln\frac{9}{2(v+3)^{3}}$$ AG

[8]

4. (a) (i) Let $\frac{1}{x^2+4x} \equiv \frac{A}{x} + \frac{B}{x+4}$, where A and B are constants.

$$\frac{1}{x^2+4x} \equiv \frac{A(x+4)}{x(x+4)} + \frac{Bx}{x(x+4)}$$ M1

$$\frac{1}{x^2+4x} \equiv \frac{Ax+4A+Bx}{x(x+4)}$$

$$1 \equiv (A+B)x + 4A$$ A1

$$1 = 4A$$

$$A = \frac{1}{4}$$ A1

$$0 = \frac{1}{4} + B$$

$$B = -\frac{1}{4}$$

$$\therefore \frac{1}{x^2+4x} \equiv \frac{1}{4x} - \frac{1}{4(x+4)}$$ A1

(ii)

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{a^2+4a}{4}$$

$$\frac{1}{a^2+4a}\,\mathrm{d}a = \frac{1}{4}\,\mathrm{d}t$$ (M1) for valid approach

$$\int \frac{1}{a^2+4a}\,\mathrm{d}a = \int \frac{1}{4}\,\mathrm{d}t$$ (A1) for correct approach

$$\therefore \frac{1}{4}\int\left(\frac{1}{a} - \frac{1}{a+4}\right)\mathrm{d}a = \int \frac{1}{4}\,\mathrm{d}t$$ (M1) for substitution

$$\int\left(\frac{1}{a} - \frac{1}{a+4}\right)\mathrm{d}a = \int \mathrm{d}t$$

$$\ln|a| - \ln|a+4| = t + C$$ A1

$$\ln\left|\frac{a}{a+4}\right| = t + C$$

$$\frac{a}{a+4} = e^{t+C}$$ (A1) for correct approach

$$a = (a+4)e^{t+C}$$

$$a = ae^{t+C} + 4e^{t+C}$$

$$a - ae^{t+C} = 4e^{t+C}$$ (M1) for valid approach

$$a(1-e^{t+C}) = 4e^{t+C}$$

$$a = \frac{4e^{t+C}}{1-e^{t+C}}$$ A1

$$\frac{4}{e^2-1}=\frac{4e^{0+C}}{1-e^{0+C}}$$ (M1) for substitution

$$\frac{4}{e^2-1}=\frac{4e^{C}}{1-e^{C}}$$

$$\frac{4}{e^2-1}=\frac{4}{e^{-C}-1}$$

$$2=-C$$

$$C=-2$$ (A1) for correct value

$$\therefore a=\frac{4e^{t-2}}{1-e^{t-2}}$$ A1

[14]

(b) $$a=\frac{4e^{t-2}}{1-e^{t-2}}$$

$$\frac{dv}{dt}=\frac{4e^{t-2}}{1-e^{t-2}}$$

$$v=\int\frac{4e^{t-2}}{1-e^{t-2}}dt$$ A1

Let $u=1-e^{t-2}$. M1

$$\frac{du}{dt}=-e^{t-2}\Rightarrow -1\cdot du=e^{t-2}dt$$

$$\therefore v=\int-\frac{4}{u}du$$ A1

$$v=-4\ln u+D$$ A1

$$v=-4\ln\left|1-e^{t-2}\right|+D$$ A1

$$8-4\ln(1-e^{-2})=-4\ln\left|1-e^{0-2}\right|+D$$ M1

$$8-4\ln(1-e^{-2})+4\ln(1-e^{-2})=D$$

$$D=8$$

$$v=-4\ln\left|1-e^{t-2}\right|+8$$ A1

$$-2\le t-2<0$$

$$e^{-2}\le e^{t-2}<1$$

$$0<1-e^{t-2}\le 1-e^{-2}<1$$ M1

$\ln(1-e^{t-2})<0$ for $0\le t<2$

$\therefore v=-4\ln\left|1-e^{t-2}\right|+8>0$ for $0\le t<2$ R1

Thus, the particle never stops in $0\le t<2$. AG

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