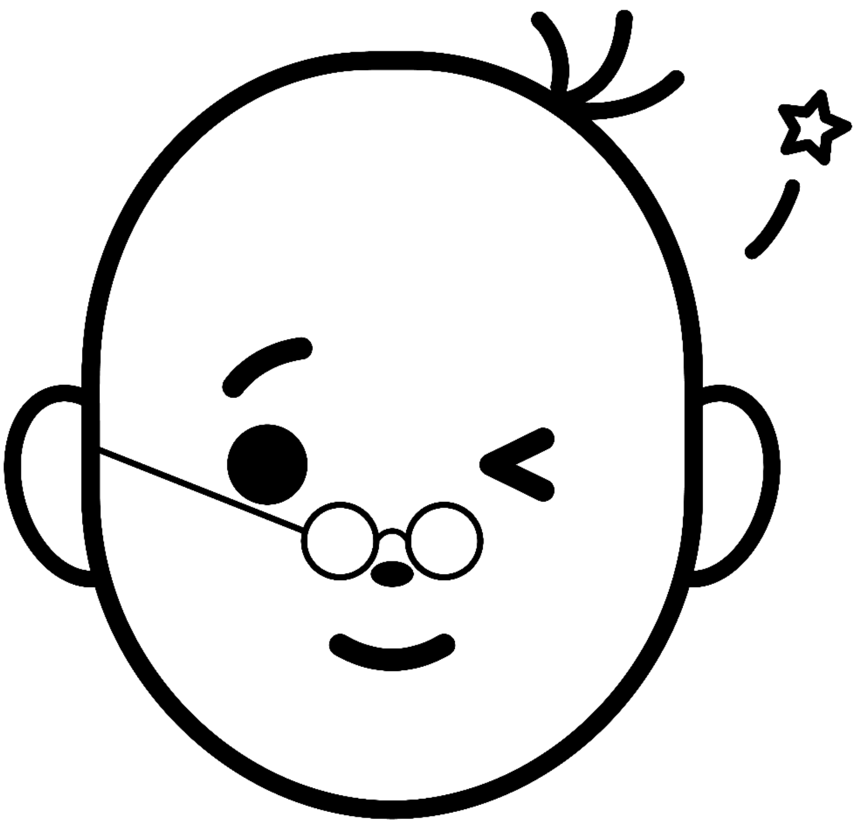
Your Intensive Notes Applications and Interpretation Higher Level for IBDP Mathematics



Functions

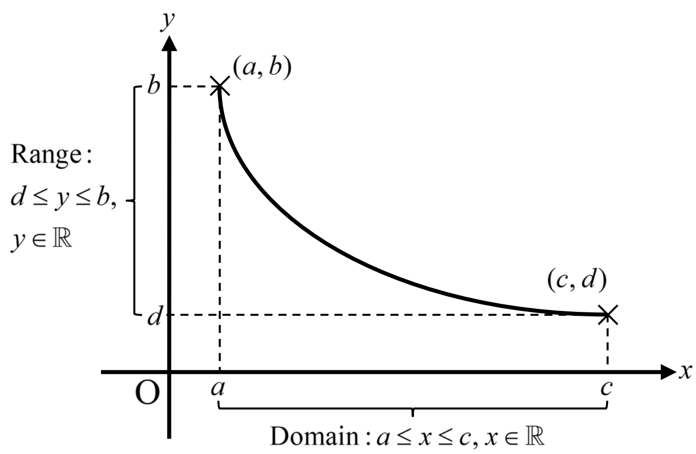
Topics Covered

1 Functions

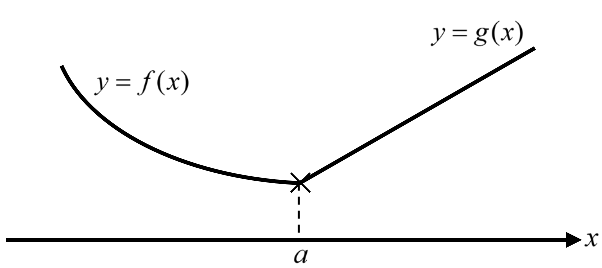
Important Notes

Notations related to a general function $y = f(x)$:

1. $f(a)$: Functional value (value of y) when $x = a$
2. Domain: Set of all possible values of x
3. Range: Set of all possible values of y

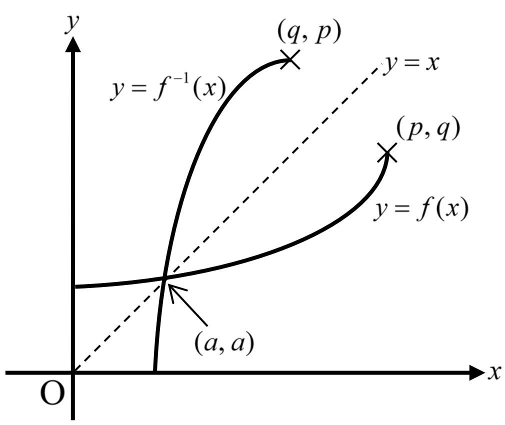


4. Root(s) of the equation $f(x) = 0$: x-intercept(s) of the graph of the corresponding function $y = f(x)$, which is equivalent to the zero(s) of $y = f(x)$
5. $(f \circ g)(x) = f(g(x))$: Composite function when $g(x)$ is substituted into $f(x)$
6. $y = ax^3 + bx^2 + cx + d$: Cubic function, where $a \neq 0$
7. $y = \begin{cases} f(x) & x \le a \\ g(x) & x > a \end{cases}$: Piecewise function



Properties of $y = f^{-1}(x)$:

1. Domain of f^{-1} is consistent with range of f
2. Range of f^{-1} is consistent with domain of f
3. $f(f^{-1}(x)) = f^{-1}(f(x)) = x$
4. Graph of $y = f^{-1}(x)$: Reflection of the graph of $y = f(x)$ about $y = x$
5. The points of intersection of the graphs of f^{-1} and f lies on $y = x$
6. $y = f^{-1}(x)$ exists only when $y = f(x)$ is one-to-one in the restricted domain

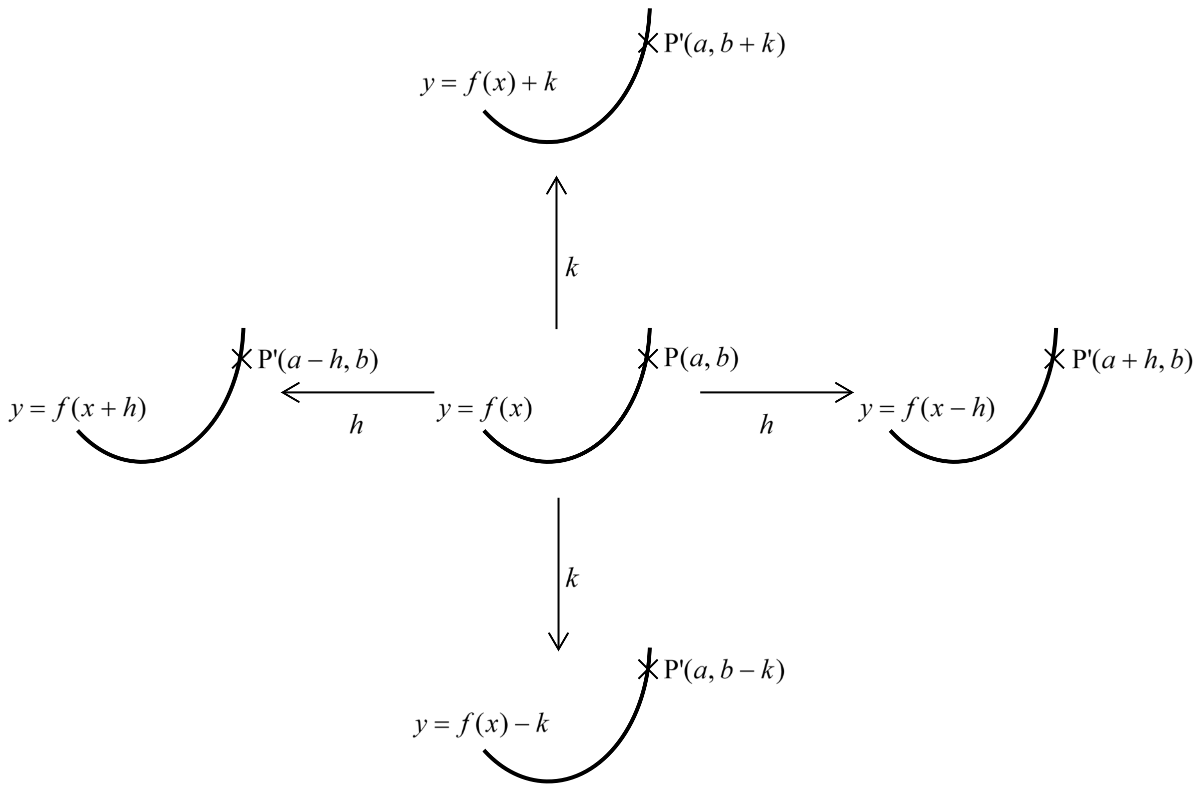


Steps of finding an expression of the inverse function $y = f^{-1}(x)$ from $y = f(x)$:

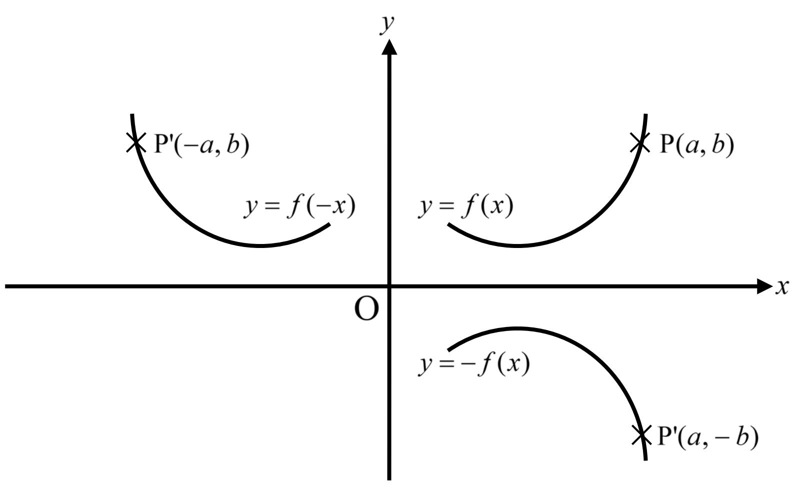
1. Start from expressing y in terms of x
2. Interchange x and y
3. Make y the subject in terms of x

Summary of the transformations of functions:

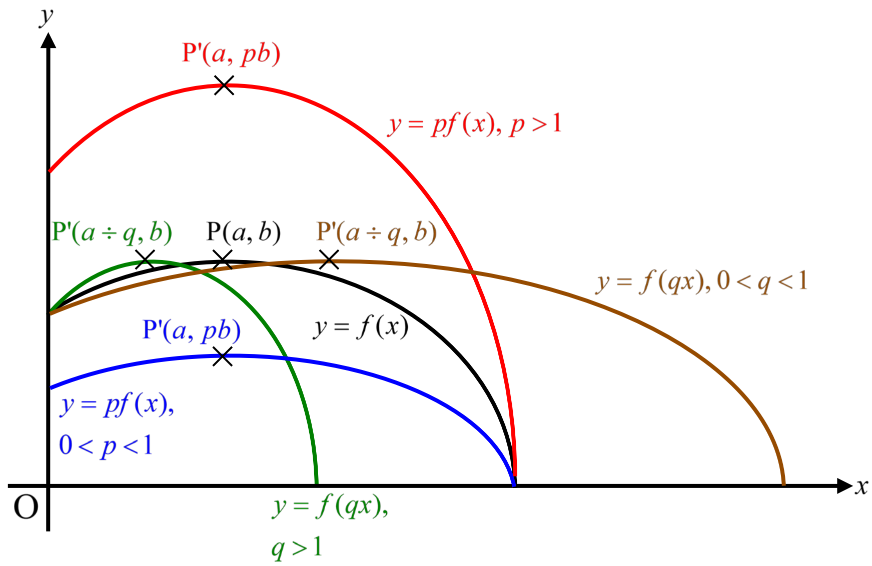
1. $y = f(x) \rightarrow y = f(x) + k$: Translate upward by k units
2. $y = f(x) \rightarrow y = f(x) - k$: Translate downward by k units
3. $y = f(x) \rightarrow y = f(x - h)$: Translate to the right by h units
4. $y = f(x) \rightarrow y = f(x + h)$: Translate to the left by h units



5. $y = f(x) \rightarrow y = -f(x)$: Reflection about the x-axis
6. $y = f(x) \rightarrow y = f(-x)$: Reflection about the y-axis



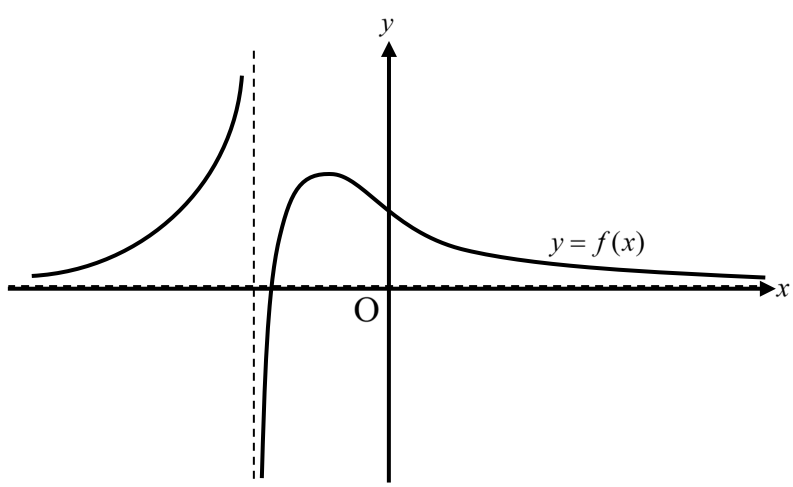
7. $y = f(x) \rightarrow y = p\,f(x)$: Vertical stretch of scale factor p, $p > 1$ (compression for $0 < p < 1$)
8. $y = f(x) \rightarrow y = f(q\,x)$: Horizontal compression of scale factor q, $q > 1$ (stretch for $0 < q < 1$)



9. $\begin{pmatrix} h \\ k \end{pmatrix}$: Composite translation vector of h units to the right and k units upward

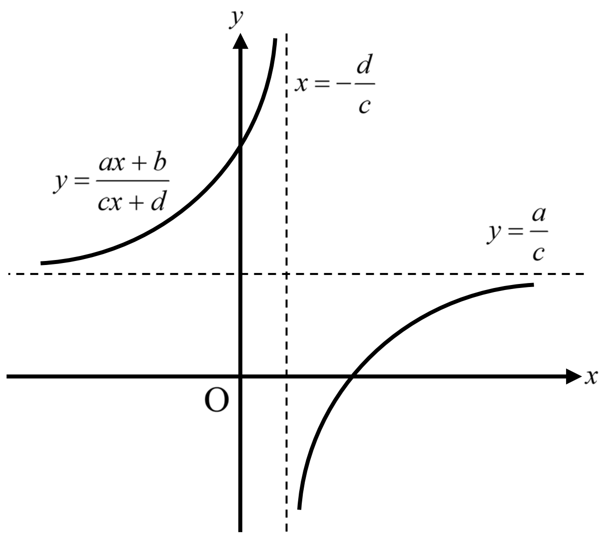
Types of asymptotes of the graph of $y = f(x)$:

1. Vertical asymptote: The vertical boundary where $f(x)$ is undefined
2. The equation of the vertical asymptote can be found by considering the denominator expression of $f(x)$ equals to zero
3. Horizontal asymptote: The horizontal boundary (level) where y approaches when x tends to positive/negative infinity
4. The equation of the horizontal asymptote can be found by considering $y = \lim_{x \to \infty} f(x)$



Properties of the rational function $y=\frac{ax+b}{cx+d}$, $a,b,c,d\in\mathbb{R}$, $c\neq 0$:

1. $y=\frac{1}{x}$: Reciprocal function
2. $y=\frac{a}{c}$: Horizontal asymptote
3. $x=-\frac{d}{c}$: Vertical asymptote from $cx+d=0$
4. Substitute $y=0$ and make x the subject to find the x-intercept
5. Substitute $x=0$ and make y the subject to find the y-intercept



Types of variations:

1. $y=kx$: y is directly proportional to x, where $k\neq 0, k\in\mathbb{R}$
2. $y=\frac{k}{x}$: y is inversely proportional to x, where $k, x\neq 0, k\in\mathbb{R}$

Notes on GDC		
TEXAS TI-84 Plus CE y= to input the function →2nd window to set the starting row to be at least 1000 →2nd graph to look at the function values when x is at least 1000 to find the equation of the horizontal asymptote of the graph	**TEXAS TI-Nspire CX** Graph to input the function to generate a table →ctrl T to generate a table →menu 2 5 to set the starting row to look at the function values when x is at least 1000 to find the equation of the horizontal asymptote of the graph	**CASIO fx-CG50** Table to input the function to generate a table →F5 to set the starting row to be at least 1000 →F6 to look at the function values when x is at least 1000 to find the equation of the horizontal asymptote of the graph

Example 2.1

The number of cells in a culture, $N(t)$, t hours after it has been established is given by $N(t)=10t^2$ for $0\le t\le 5$.

(a) Write down

(i) the initial number of cells;

[1]

$N(0)=0$ (A1)

(ii) $N(4)$;

[1]

$N(4)=160$ (A1)

(iii) $N^{-1}(90)$.

[1]

$N^{-1}(90)=3$ (A1)

(b) Find the range of N.

[2]

$N(5)$

$=10(5)^2$ $N(5)$ (M1)

$=250$

Thus, the range of N is $0\le N\le 250$, $N\in\mathbb{R}$. (A1)

(c) In the context of this question, interpret the meaning of $N^{-1}(40)=2$.

[1]

There are 40 in a culture, 2 hours after it has been established. (A1)

(d) Find an expression of $N^{-1}(t)$.

[2]

$N=10t^2$

$\rightarrow t=10N^2$ Interchange t and N (M1)

$0.1t=N^2$

$N=\sqrt{0.1t}$

$\therefore N^{-1}(t)=\sqrt{0.1t}$ (A1)

Further observation suggests that the model shall be modified by using a series of two transformations. Let $Q(t)$ be the modified number of cells in a culture, where $Q(t) = N(1.1t) + 20$ for $0 \le t \le 5$.

(e) Give a full geometric description of each of the two transformations.

[2]

Horizontal compression of scale factor 1.1 (A1)
followed by an upward translation by 20 units (A1)

(f) Find an expression of $Q(t)$.

[2]

$Q(t)$
$= N(1.1t) + 20$
$= 10(1.1t)^2 + 20$ $\quad$ $10(1.1t)^2 + 20$ (M1)
$= 12.1t^2 + 20$ $\quad$ (A1)

Exercise 2.1

Transfers from the airport to a passenger's living place have various prices. The price $P(a)$ dollars of the journey when the passenger lives a kilometres from the airport is given by $P(a)=0.8a^2+50$, where $0\le a\le 50$.

(a) Write down

(i) $P(0)$;

[1]

(ii) $P^{-1}(70)$;

[1]

(b) Find the range of P.

[2]

(c) In the context of this question, interpret the meaning of $P^{-1}(370)=20$.

[1]

(d) Find an expression of $P^{-1}(a)$.

[2]

Further observation suggests that the model to be used in the next year shall be modified by using a series of two transformations. Let $Q(a)$ be the modified of the journey price when the passenger lives a kilometres, where $Q(a) = 1.2P(a) + 5$ for $0 \le a \le 50$.

(e) Give a full geometric description of each of the two transformations. [2]

(f) Find an expression of $Q(a)$. [2]

Example 2.2

Let $\$C$ be the cost of manufacturing a cubical block of side x cm. It is given that C is directly proportional to the square root of x, and the cost of manufacturing a cubical block of side 9 cm is $\$36$.

(a) Express C in terms of x.

[2]

Let $C = k\sqrt{x}$, where $k \neq 0$. — $C = k\sqrt{x}$ (M1)

$36 = k\sqrt{9}$

$k = 12$

$\therefore C = 12\sqrt{x}$ — (A1)

(b) Write down the cost of manufacturing a cubical block of side 16 cm.

[1]

$\$48$ — (A1)

Suppose that the extra cost $\$24$ is taken into account.

(c) Find the length of the side of a cubical block with cost $\$48$.

[2]

$48 = 12\sqrt{x} + 24$ — Correct equation (A1)

$24 = 12\sqrt{x}$

$2 = \sqrt{x}$

$x = 4$

Thus, the required length is 4 cm. — (A1)

The cost factor r is defined as $r = 10 + C^2$.

(d) Express r in terms of x.

[2]

r

$= 10 + C^2$

$= 10 + (12\sqrt{x})^2$ — $10 + (12\sqrt{x})^2$ (M1)

$= 10 + 144x$ — (A1)

Exercise 2.2

Let $\$P$ be the price of a tetrahedron model of surface area of $A\text{ cm}^2$. It is given that P is inversely proportional to A. When $A = 16$, $P = 15$.

(a) Express P in terms of A.

[2]

(b) Write down the price of a tetrahedron model of surface area of 80 cm^2.

[1]

(c) Interpret the condition on the price of a tetrahedron model of a large surface area.

[1]

The price factor α is defined as $\alpha = \dfrac{14400}{P^2}$.

(d) Express α in terms of A.

[2]

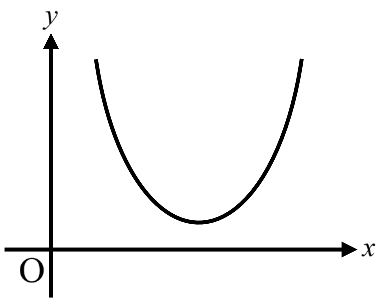
2 Quadratic Functions

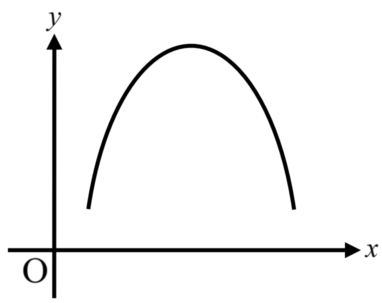
Important Notes

Quadratic function: A polynomial function with the greatest power of x equals to 2

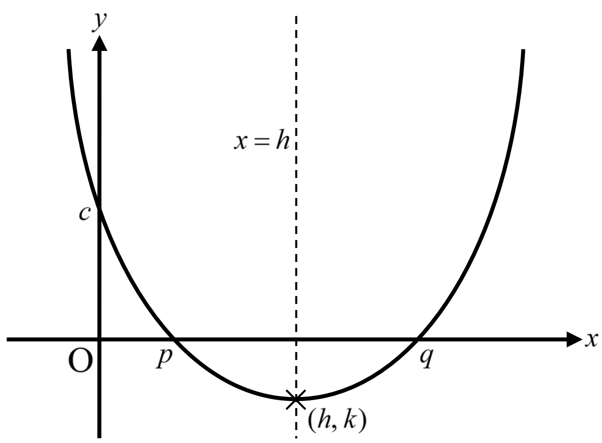
Properties of a quadratic function in its general form $y = ax^2 + bx + c$, $a \neq 0$

1. $a > 0$: Opens upward; $a < 0$: Opens downward





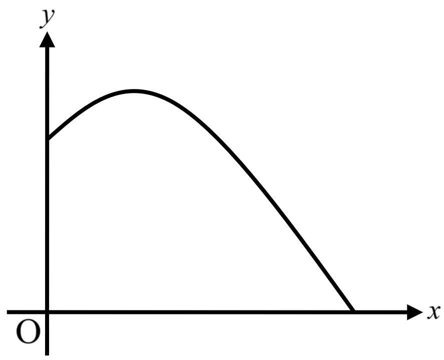
2. c: y-intercept of the graph
3. $x = h$: Equation of the axis of symmetry of the graph
4. $h = -\dfrac{b}{2a} = \dfrac{p+q}{2}$: x-coordinate of the vertex of the graph
5. $k = ah^2 + bh + c$: y-coordinate of the vertex of the graph, which is also the extreme (maximum when $a < 0$/minimum when $a > 0$) value of y



Root(s) of the quadratic equation $ax^2 + bx + c = 0$: x-intercept(s) of the graph of the corresponding quadratic function $y = ax^2 + bx + c$

Example 2.3

A ball is kicked from the top of a vertical cliff onto a horizontal grass ground. The path of the ball can be modelled by the quadratic curve $y=-x^2+4x+20$, where x m and y m are the horizontal distance from the cliff and the vertical distance above the ground respectively, as shown in the diagram below.



(a) Write down the vertical height of the cliff.

[1]

20 m (A1)

(b) Find the maximum height of the trajectory of the ball.

[2]

By considering the graph of $y=-x^2+4x+20$, the coordinates of the maximum point are $(2, 24)$. GDC approach (M1)

$\therefore$ The required maximum height is 24 m. (A1)

(c) Write down the horizontal distance of the ball from the cliff when the ball is at the same vertical level when the ball is first kicked.

[1]

4 m (A1)

(d) Find the horizontal distance from the cliff to the position at which the ball hits the grass ground.

[3]

$-x^2+4x+20=0$ Correct equation (A1)

By considering the graph of $y=-x^2+4x+20$, the horizontal intercept is 6.8989795. GDC approach (M1)

$\therefore$ The required horizontal distance is 6.90 m. (A1)

(e) State, for this model,

(i) an appropriate domain for x;

[1]

$0 \le x \le 6.90$, $x \in \mathbb{R}$ (A1)

(ii) an appropriate range for y.

[1]

$0 \le y \le 24$, $y \in \mathbb{R}$ (A1)

(f) Write down one possible limitation of using $y = -x^2 + 4x + 20$ to model the path of the ball.

[1]

The model does not consider air resistance. (R1)

Exercise 2.3

In a right-angled triangle, the lengths of the two shorter sides are $(x-18)$ cm and $(x-1)$ cm respectively. The area A cm^2 of the triangle is given by $A = 0.5x^2 - 9.5x + 9$.

(a) Write down the area of the triangle when $x = 20$.

[1]

(b) State, for this model,

(i) an appropriate domain for x;

[1]

(ii) an appropriate range for A.

[1]

Consider the case when the area of the triangle is 55 cm^2.

(c) (i) Find x.

[3]

(ii) Hence, find the corresponding perimeter.

[3]

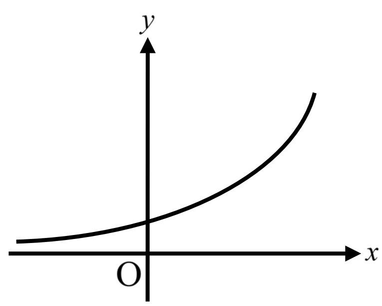
3 Exponential and Logarithmic Functions

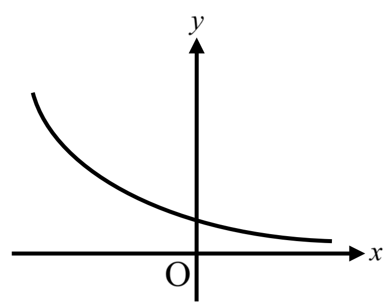
Important Notes

Exponential function: A function with x to be the power (exponent) of a positive real number other than 1

Properties of an exponential function in the form $y=a^x$, base $a\in\mathbb{R}^+$

1. $a>1$: Exponentially increase — $0<a<1$: Exponentially decrease

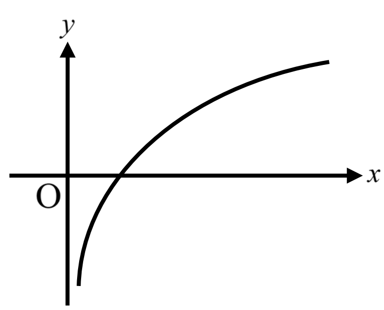


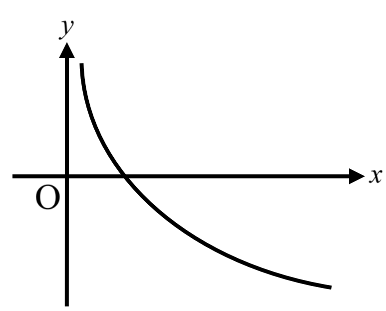


2. $a^0=1$: y-intercept of the graph
3. $x\in\mathbb{R}$: Domain of $y=a^x$
4. $y>0$, $y\in\mathbb{R}$: Range of $y=a^x$
5. $y=0$: Equation of the horizontal asymptote of the graph

Properties of a logarithmic function in the form $y=\log_a x$, base $a\in\mathbb{R}^+$

1. $y=\log_a x$ is the inverse function of $y=a^x$
2. $a>1$: Increase — $0<a<1$: Decrease





3. 1: x-intercept of the graph
4. $x>0$, $x\in\mathbb{R}$: Domain of $y=\log_a x$
5. $y\in\mathbb{R}$: Range of $y=\log_a x$
6. $x=0$: Equation of the vertical asymptote of the graph

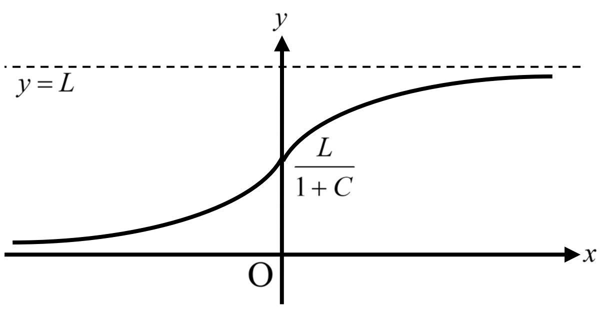
7. $y=\log x(=\log_{10} x)$: Logarithmic function of the common base (base 10)

8. $y=\ln x(=\log_e x)$: Natural logarithmic function of the base e, where $e=2.718281828\cdots$ is the exponential number

Laws of logarithm, where $a, b, c, p, q, x>0$:

1. $b=a^x \Leftrightarrow x=\log_a b$
2. $1=a^0 \Leftrightarrow 0=\log_a 1$
3. $a=a^1 \Leftrightarrow 1=\log_a a$
4. $\log_a p+\log_a q=\log_a (pq)$
5. $\log_a p-\log_a q=\log_a \left(\frac{p}{q}\right)$
6. $\log_a p^n=n\log_a p$

$f(x)=\frac{L}{1+Ce^{-kx}}$: Logistic function, where $L, C, k\in\mathbb{R}^+$



Semi-log model:

1. $y=k\cdot a^x \Leftrightarrow \ln y=(\ln a)\,x+\ln k$: Semi-log model
2. $\ln a$: Slope (Gradient) of the straight line graph on a $\ln y$-x plane
3. $\ln k$: Vertical intercept of the straight line graph on a $\ln y$-x plane

Log-log model:

1. $y=k\cdot x^n \Leftrightarrow \ln y=n\ln x+\ln k$: Log-log model
2. n: Slope (Gradient) of the straight line graph on a $\ln y$-$\ln x$ plane
3. $\ln k$: Vertical intercept of the straight line graph on a $\ln y$-$\ln x$ plane

Example 2.4

A town is concerned about pollution, and decides to look at the number of people using private cars. At the end of 2023, there were 500 private cars in the town. After t years the number of private cars, N, in the city is given by $N = N_0 e^{kt}$, N_0, $k > 0$, $t \geq 0$.

(a) Show that $N_0 = 500$.

[1]

$500 = N_0 e^{k(0)}$ — $N = 500$ & $t = 0$ (A1)

$N_0 = 500$ — (AG)

There are 710 private cars at the end of 2026.

(b) Find k.

[3]

$710 = 500e^{k(3)}$ — $N = 710$ & $t = 3$ (A1)

$500e^{3k} - 710 = 0$

By considering the graph of $y = 500e^{3k} - 710$,

the horizontal intercept is 0.1168856. — GDC approach (M1)

$\therefore k = 0.117$ — (A1)

(c) Find the year in which the number of private cars is triple the number of private cars there were at the end of 2023.

[3]

$500(3) = 500e^{0.1168856t}$ — Correct equation (A1)

$3 = e^{0.1168856t}$

$e^{0.1168856t} - 3 = 0$

By considering the graph of $y = e^{0.1168856t} - 3$,

the horizontal intercept is 9.3990388. — GDC approach (M1)

$\therefore$ The required year is 2033. — (A1)

Another study suggests that the relationship between N and t shall be defined by an alternative model $t = -49.7 + 18.3\log_{10} N$.

(d) Find, under the new model, the year in which the number of private cars is triple the number of private cars there were at the end of 2023.

[2]

t

$= -49.7 + 18.3\log_{10}(500(3))$ $N = 500(3)$ (M1)

$= 8.422470041$

$\therefore$ The required year is 2032. (A1)

Exercise 2.4

A population of Bulbul birds, P, can be modelled by the equation $P = P_0 e^{kt}$, where P_0 is the initial population of Bulbul birds and t is measured in decades. After one decade (ten years), it is estimated that the population is 10% less than the initial population.

(a) Find k, correct the answer to four decimal places.

[3]

(b) Hence, interpret the meaning of the value of k.

[1]

(c) Find the least number of complete years such that the population of Bulbul birds is half of the initial population.

[3]

A population of Zebra finches, Q, can be modelled by the equation $t = 71 - 18.8\log_{10} Q$, where t is measured in decades.

(d) Find, under the new model, the least number of complete years such that the population of Zebra finches reaches 3000 for the first time.

[2]

Example 2.5

The relationship between the amount D, in milligrams, of a medicinal drug in the body t hours after it was injected is given by $\ln D = \ln 25 + (\ln 0.87)t$, $t \geq 0$. The graph of $\ln D$ versus t is sketched which is shown in the form of a straight line.

(a) Write down, correct to four decimal places, for this straight line,

(i) the slope;

[1]

The slope
$= \ln 0.87$
$= -0.1392620673$
$= -0.1393$ (A1)

(ii) the vertical intercept.

[1]

The vertical intercept
$= \ln 25$
$= 3.218875825$
$= 3.2189$ (A1)

(b) Express D in terms of t.

[3]

$\ln D = \ln 25 + (\ln 0.87)t$
$\ln D = \ln 25 + \ln 0.87^t$ $n \ln a = \ln a^n$ (M1)
$\ln D = \ln(25 \cdot 0.87^t)$ $\ln a + \ln b = \ln(ab)$ (M1)
$D = 25 \cdot 0.87^t$ (A1)

(c) Hence,

(i) write down the initial dose of the drug;

[1]

25 mg (A1)

(ii) find the percentage of the drug that leaves the body each hour.

[2]

0.87
$= 1 - 0.13$ $1 - 0.13$ (M1)
$= 1 - 13\%$
Thus, the required percentage is 13%. (A1)

(d) Find the amount of the drug remaining in the body eight hours after the injection.

[2]

The required amount

$= 25 \cdot 0.87^8$ $t = 8$ (M1)

$= 8.205291789$ mg

$= 8.21$ mg (A1)

Exercise 2.5

Pamela carries out an experiment on the growth of mould. She believes that the relationship between the area A covered by mould in mm^2 and the time t in days since the start of the growth can be modelled by $\ln A = \ln 128 + 0.864 \ln t$. The graph of $\ln A$ versus $\ln t$ is sketched which is shown in the form of a straight line.

(a) Write down, for this straight line,

(i) the slope; [1]

(ii) the vertical intercept, correct the answer to five decimal places. [1]

(b) Express A in terms of t. [3]

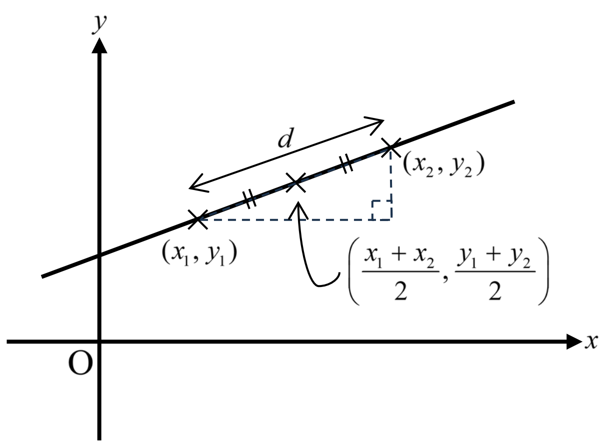
(c) Hence, find the area covered by mould one week since the start of the growth. [2]

4 Equations of Straight Lines

Important Notes

Consider any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on a $x-y$ plane:

1. $m = \frac{y_2 - y_1}{x_2 - x_1}$: Slope (gradient) of PQ
2. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$: Distance between P and Q
3. $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$: The mid-point of PQ



Consider any two straight lines L_1 and L_2 with corresponding slopes m_1 and m_2 respectively:

1. $m_1 = m_2$ if L_1 and L_2 are parallel ($L_1 // L_2$)
2. $m_1 \times m_2 = -1$ if L_1 and L_2 are perpendicular ($L_1 \perp L_2$)

$y - y_1 = m(x - x_1)$: The point-slope formula to find the equation of a straight line with slope m and a fixed point (x_1, y_1) on the line

Forms of equations of straight lines:

1. $y = mx + c$: Slope-intercept form with slope m and y-intercept c
2. $Ax + By + C = 0$: General form, where $A \in \mathbb{Z}^+$, $B, C \in \mathbb{Z}$

Axes intercepts of a straight line:

1. Substitute $y = 0$ and make x the subject to find the x-intercept
2. Substitute $x = 0$ and make y the subject to find the y-intercept

Example 2.6

A line joins the points $\mathrm{A}(10, 3)$ and $\mathrm{B}(-2, -7)$.

(a) Find the gradient of the line AB.

[2]

The gradient

$= \frac{-7-3}{-2-10}$ $\quad m = \frac{y_2 - y_1}{x_2 - x_1}$ (M1)

$= \frac{5}{6}$ (A1)

Let M be the midpoint of the line AB.

(b) (i) Write down the coordinates of M.

[1]

$(4, -2)$ (A1)

(ii) Hence, find the exact distance between A and M.

[2]

The exact distance

$= \sqrt{(4-10)^2 + (-2-3)^2}$ $\quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (M1)

$= \sqrt{61}$ (A1)

(c) Find the equation of the line perpendicular to AB and passing through M, giving the answer in slope-intercept form.

[3]

The required slope

$= -1 \div \frac{5}{6}$ $\quad m_1 \times m_2 = -1$ (M1)

$= -\frac{6}{5}$

The equation:

$y - (-2) = -\frac{6}{5}(x - 4)$ $\quad y - y_1 = m(x - x_1)$ (M1)

$y + 2 = -\frac{6}{5}x + \frac{24}{5}$

$y = -\frac{6}{5}x + \frac{14}{5}$ (A1)

Exercise 2.6

A line joins the points $A(0, -9)$ and $B(-8, 1)$.

(a) Find the gradient of the line AB.

[2]

Let M be the midpoint of the line AB.

(b) (i) Write down the coordinates of M.

[1]

(ii) Hence, find the exact distance between B and M.

[2]

(c) Find the equation of the line perpendicular to AB and passing through B, giving the answer in general form.

[3]