Chapter 13 Solution

Exercise 55

1. The area of R

$= \int_{\frac{1}{6}}^{\frac{1}{2}} f(x)\mathrm{d}x$ (A1) for correct approach

$= \int_{\frac{1}{6}}^{\frac{1}{2}} \pi \operatorname{cosec}^2 \pi x \mathrm{d}x$

Let $u = \pi x$. (M1) for substitution

$\frac{\mathrm{d}u}{\mathrm{d}x} = \pi \Rightarrow \mathrm{d}u = \pi \mathrm{d}x$

$x = \frac{1}{2} \Rightarrow u = \pi\left(\frac{1}{2}\right) = \frac{\pi}{2}$

$x = \frac{1}{6} \Rightarrow u = \pi\left(\frac{1}{6}\right) = \frac{\pi}{6}$

$\therefore \int_{\frac{1}{6}}^{\frac{1}{2}} f(x)\mathrm{d}x = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^2 u \mathrm{d}u$ (A2) for correct working

$\int_{\frac{1}{6}}^{\frac{1}{2}} f(x)\mathrm{d}x = \left[-\cot u\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ A1

$\int_{\frac{1}{6}}^{\frac{1}{2}} f(x)\mathrm{d}x = -\cot\frac{\pi}{2} - \left(-\cot\frac{\pi}{6}\right)$

$\int_{\frac{1}{6}}^{\frac{1}{2}} f(x)\mathrm{d}x = 0 - (-\sqrt{3})$

$\int_{\frac{1}{6}}^{\frac{1}{2}} f(x)\mathrm{d}x = \sqrt{3}$ A1

[6]

2. The area of R

$= \int_1^{e^4} f(x)\mathrm{d}x$ (A1) for correct approach

$= \int_1^{e^4} \ln x\mathrm{d}x$

$= \left[(\ln x)(x)\right]_1^{e^4} - \int_1^{e^4} x\mathrm{d}(\ln x)$ (A2) for correct working

$= \left[x\ln x\right]_1^{e^4} - \int_1^{e^4} x \cdot \frac{1}{x}\mathrm{d}x$

$= \left[x\ln x\right]_1^{e^4} - \int_1^{e^4} 1\mathrm{d}x$ (M1) for valid approach

$= \left[x\ln x\right]_1^{e^4} - \left[x\right]_1^{e^4}$ A1

$= e^4 \ln e^4 - 1\ln 1 - (e^4 - 1)$

$= 4e^4 - e^4 + 1$

$= 3e^4 + 1$ A1

[6]

3. $\int_{\sqrt{\frac{\pi}{6}}}^{\frac{\sqrt{\pi}}{k}} f(x)\,dx = 2-\sqrt{2}$ (M1) for setting equation

$\int_{\sqrt{\frac{\pi}{6}}}^{\frac{\sqrt{\pi}}{k}} 2x\cot x^2 \operatorname{cosec} x^2\,dx = 2-\sqrt{2}$

Let $u = x^2$. (M1) for substitution

$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$

$x = \frac{\sqrt{\pi}}{k} \Rightarrow u = \left(\frac{\sqrt{\pi}}{k}\right)^2 = \frac{\pi}{k^2}$

$x = \sqrt{\frac{\pi}{6}} \Rightarrow u = \left(\sqrt{\frac{\pi}{6}}\right)^2 = \frac{\pi}{6}$

$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{k^2}} \cot u \operatorname{cosec} u\,du = 2-\sqrt{2}$ (A2) for correct working

$\left[-\operatorname{cosec} u\right]_{\frac{\pi}{6}}^{\frac{\pi}{k^2}} = 2-\sqrt{2}$ A1

$-\operatorname{cosec}\frac{\pi}{k^2} - \left(-\operatorname{cosec}\frac{\pi}{6}\right) = 2-\sqrt{2}$

$-\operatorname{cosec}\frac{\pi}{k^2} - (-2) = 2-\sqrt{2}$

$\operatorname{cosec}\frac{\pi}{k^2} = \sqrt{2}$

$\therefore \sin\frac{\pi}{k^2} = \frac{1}{\sqrt{2}}$ (A1) for correct approach

$\frac{\pi}{k^2} = \frac{\pi}{4}$

$k^2 = 4$

$k = 2$ or $k = -2$ (*Rejected*) A1

[7]

4.	$\int_2^k f(x)\mathrm{d}x = e^2(2e-1)$	(M1) for setting equation
	$\int_2^k xe^x\mathrm{d}x = e^2(2e-1)$	
	Let $\theta = e^x$.	(M1) for valid approach
	$\frac{\mathrm{d}\theta}{\mathrm{d}x} = e^x \Rightarrow \mathrm{d}\theta = e^x\mathrm{d}x$	
	$\therefore \int_2^k xe^x\mathrm{d}x = \int_2^k x\mathrm{d}(e^x)$	
	$\left[(x)(e^x)\right]_2^k - \int_2^k e^x\mathrm{d}x = e^2(2e-1)$	(A2) for correct working
	$\left[xe^x\right]_2^k - \left[e^x\right]_2^k = e^2(2e-1)$	A1
	$ke^k - 2e^2 - (e^k - e^2) = 2e^3 - e^2$	
	$(k-1)e^k - e^2 = 2e^3 - e^2$	
	$(k-1)e^k = 2e^3$	
	$\therefore k = 3$	A1

[6]

Exercise 56

1. The volume of the solid generated

$= \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \pi y^2 \mathrm{d}x$ (A1) for correct approach

$= \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \pi \cot\left(x - \frac{\pi}{2}\right) \operatorname{cosec}\left(x - \frac{\pi}{2}\right) \mathrm{d}x$

Let $u = x - \frac{\pi}{2}$. (M1) for substitution

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \Rightarrow \mathrm{d}u = \mathrm{d}x$

$x = \frac{5\pi}{6} \Rightarrow u = \frac{5\pi}{6} - \frac{\pi}{2} = \frac{\pi}{3}$

$x = \frac{2\pi}{3} \Rightarrow u = \frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$

$\therefore \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \pi y^2 \mathrm{d}x = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \pi \cot u \operatorname{cosec} u \mathrm{d}u$ (A2) for correct working

$\int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \pi y^2 \mathrm{d}x = \left[-\pi \operatorname{cosec} u\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$ A1

$\int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \pi y^2 \mathrm{d}x = -\pi \operatorname{cosec} \frac{\pi}{3} - \left(-\pi \operatorname{cosec} \frac{\pi}{6}\right)$

$\int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \pi y^2 \mathrm{d}x = -\frac{2\sqrt{3}\pi}{3} + 2\pi$

$\int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \pi y^2 \mathrm{d}x = \frac{2(3 - \sqrt{3})\pi}{3}$ A1

[6]

2. $y = \ln \sqrt[\pi]{x} - \ln \sqrt[\pi]{\pi}$

$y = \ln \sqrt[\pi]{\dfrac{x}{\pi}}$

$y = \ln \left(\dfrac{x}{\pi}\right)^{\frac{1}{\pi}}$ (M1) for valid approach

$y = \dfrac{1}{\pi} \ln \left(\dfrac{x}{\pi}\right)$

$\pi y = \ln \left(\dfrac{x}{\pi}\right)$

$e^{\pi y} = \dfrac{x}{\pi}$

$\therefore x = \pi e^{\pi y}$ A1

The volume of the solid generated

$= \int_0^{2\pi} \pi x^2 \mathrm{d}y$ (A1) for correct approach

$= \int_0^{2\pi} \pi^2 e^{2\pi y} \mathrm{d}y$

Let $u = 2\pi y$. (M1) for substitution

$\dfrac{\mathrm{d}u}{\mathrm{d}y} = 2\pi \Rightarrow \dfrac{1}{2}\mathrm{d}u = \pi \mathrm{d}y$

$y = 2\pi \Rightarrow u = 2\pi(2\pi) = 4\pi^2$

$y = 0 \Rightarrow u = 2\pi(0) = 0$

$\therefore \int_0^{2\pi} \pi x^2 \mathrm{d}y = \int_0^{4\pi^2} \dfrac{\pi}{2} e^u \mathrm{d}u$ (A2) for correct working

$\int_0^{2\pi} \pi x^2 \mathrm{d}y = \left[\dfrac{\pi}{2} e^u\right]_0^{4\pi^2}$ A1

$\int_0^{2\pi} \pi x^2 \mathrm{d}y = \dfrac{\pi}{2} e^{4\pi^2} - \dfrac{\pi}{2} e^0$

$\int_0^{2\pi} \pi x^2 \mathrm{d}y = \dfrac{\pi}{2}(e^{4\pi^2} - 1)$ A1

[8]

3. $\dfrac{x^2}{9a^2}+\dfrac{y^2}{4a^2}=1$

$\dfrac{y^2}{4a^2}=1-\dfrac{x^2}{9a^2}$

$y^2=4a^2-\dfrac{4}{9}x^2$ A1

When $y=0$,

$\dfrac{x^2}{9a^2}+0=1$ M1

$x^2=9a^2$

$x=-3a$ or $x=3a$ A1

The volume of the rugby model

$=\int_{-3a}^{3a}\pi y^2 \mathrm{d}x$ A1

$=\int_{-3a}^{3a}\pi\left(4a^2-\dfrac{4}{9}x^2\right)\mathrm{d}x$

$=\pi\left[4a^2x-\dfrac{4}{9}\left(\dfrac{1}{3}x^3\right)\right]_{-3a}^{3a}$ A1

$=\pi\left[4a^2x-\dfrac{4}{27}x^3\right]_{-3a}^{3a}$

$=\pi\left(\left(4a^2(3a)-\dfrac{4}{27}(3a)^3\right)-\left(4a^2(-3a)-\dfrac{4}{27}(-3a)^3\right)\right)$ A1

$=\pi(12a^3-4a^3+12a^3-4a^3)$

$=16\pi a^3$ AG

[6]

4. $hx + ry - 3hr = 0$

$hx = 3hr - ry$

$x = 3r - \frac{r}{h}y$ A1

The volume of the conical frustum

$= \int_0^{2h} \pi x^2 \mathrm{d}y$ A1

$= \int_0^{2h} \pi \left(3r - \frac{r}{h}y\right)^2 \mathrm{d}y$

$= \int_0^{2h} \pi \left(9r^2 - \frac{6r^2}{h}y + \frac{r^2}{h^2}y^2\right) \mathrm{d}y$ M1

$= \pi \left[9r^2 y - \frac{6r^2}{h}\left(\frac{1}{2}y^2\right) + \frac{r^2}{h^2}\left(\frac{1}{3}y^3\right)\right]_0^{2h}$ A1

$= \pi \left[9r^2 y - \frac{3r^2}{h}y^2 + \frac{r^2}{3h^2}y^3\right]_0^{2h}$ M1

$= \pi \left(\left(9r^2(2h) - \frac{3r^2}{h}(2h)^2 + \frac{r^2}{3h^2}(2h)^3\right) - 0\right)$ A1

$= \pi \left(18r^2 h - 12r^2 h + \frac{8}{3}r^2 h\right)$ A1

$= \frac{26}{3}\pi r^2 h$ AG

[7]

Exercise 57

1. (a) $\frac{2\pi}{\alpha} = 8\pi$ M1A1

$\alpha = \frac{1}{4}$ AG

[2]

(b) $f(-x) = \sec\frac{1}{4}(-x)$ M1

$f(-x) = \sec\left(-\frac{1}{4}x\right)$

$f(-x) = \frac{1}{\cos\left(-\frac{1}{4}x\right)}$

$f(-x) = \frac{1}{\cos\frac{1}{4}x}$ A1

$f(-x) = \sec\frac{1}{4}x$

$f(-x) = f(x)$

Thus, f is an even function. AG

[2]

(c) (i) $\{x : 0 \le x < 2\pi\}$ A2

(ii) $y = \sec\frac{1}{4}x$

$\Rightarrow x = \sec\frac{1}{4}y$ (M1) for swapping variables

$\frac{1}{x} = \cos\frac{1}{4}y$

$\arccos\frac{1}{x} = \frac{1}{4}y$ A1

$y = 4\arccos\frac{1}{x}$

$\therefore f^{-1}(x) = 4\arccos\frac{1}{x}$ A1

[5]

(d)	$\int_{-\pi}^{a} \pi \sec^2 \frac{1}{4}x \mathrm{d}x = 4\pi(\sqrt{3}+1)$	(A1) for correct approach
	Let $u = \frac{1}{4}x$.	(M1) for substitution
	$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{4} \Rightarrow 4\mathrm{d}u = \mathrm{d}x$	
	$x = a \Rightarrow u = \frac{1}{4}a$	
	$x = -\pi \Rightarrow u = \frac{1}{4}(-\pi) = -\frac{1}{4}\pi$	
	$\therefore \int_{-\frac{1}{4}\pi}^{\frac{1}{4}a} 4\pi \sec^2 u \mathrm{d}u = 4\pi(\sqrt{3}+1)$	(A2) for correct working
	$\therefore \int_{-\frac{1}{4}\pi}^{\frac{1}{4}a} \sec^2 u \mathrm{d}u = \sqrt{3}+1$	
	$\left[\tan u\right]_{-\frac{1}{4}\pi}^{\frac{1}{4}a} = \sqrt{3}+1$	A1
	$\tan\frac{1}{4}a - \tan\left(-\frac{1}{4}\pi\right) = \sqrt{3}+1$	
	$\tan\frac{1}{4}a - (-1) = \sqrt{3}+1$	A1
	$\tan\frac{1}{4}a = \sqrt{3}$	
	$\frac{1}{4}a = \frac{\pi}{3}$	M1
	$a = \frac{4\pi}{3}$	A1

[8]

2. (a) $f(-x)=\dfrac{1}{\sqrt{a^2-(-x)^2}}$ M1

$f(-x)=\dfrac{1}{\sqrt{a^2-x^2}}$ A1

$f(-x)=f(x)$

Thus, f is an even function. AG

[2]

(b) (i) $\{x: 0 \le x < a\}$ A2

(ii) $y=\dfrac{1}{\sqrt{a^2-x^2}}$

$\Rightarrow x=\dfrac{1}{\sqrt{a^2-y^2}}$ (M1) for swapping variables

$x^2=\dfrac{1}{a^2-y^2}$

$a^2-y^2=\dfrac{1}{x^2}$ A1

$y^2=a^2-\dfrac{1}{x^2}$

$y=\sqrt{a^2-\dfrac{1}{x^2}}$

$\therefore f^{-1}(x)=\sqrt{a^2-\dfrac{1}{x^2}}$ A1

[5]

(c) The area of R

$=\displaystyle\int_{-\frac{\sqrt{3}}{2}a}^{\frac{\sqrt{3}}{2}a}\frac{1}{\sqrt{a^2-x^2}}\,dx$ (A1) for correct approach

$=\left[\arcsin\dfrac{x}{a}\right]_{-\frac{\sqrt{3}}{2}a}^{\frac{\sqrt{3}}{2}a}$ A1

$=\arcsin\dfrac{\sqrt{3}}{2}-\arcsin\left(-\dfrac{\sqrt{3}}{2}\right)$ (M1) for substitution

$=\dfrac{\pi}{3}-\left(-\dfrac{\pi}{3}\right)$ A1

$=\dfrac{2\pi}{3}$ A1

[5]

(d) The volume of the solid generated

$$=\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a}\pi y^2\mathrm{d}x$$

$$=\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a}\pi\left(\frac{1}{\sqrt{a^2-x^2}}\right)^2\mathrm{d}x$$ A1

$$=\pi\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a}\frac{1}{a^2-x^2}\mathrm{d}x$$

Let $x=a\sin\theta$. A1

$$\frac{\mathrm{d}x}{\mathrm{d}\theta}=a\cos\theta\Rightarrow\mathrm{d}x=a\cos\theta\mathrm{d}\theta$$

$$x=\frac{\sqrt{2}}{2}a\Rightarrow\sin\theta=\frac{\sqrt{2}}{2}\Rightarrow\theta=\frac{\pi}{4}$$ M1

$$x=-\frac{\sqrt{2}}{2}a\Rightarrow\sin\theta=-\frac{\sqrt{2}}{2}\Rightarrow\theta=-\frac{\pi}{4}$$

$$\therefore\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a}\pi y^2\mathrm{d}x=\pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{a^2-a^2\sin^2\theta}\cdot a\cos\theta\mathrm{d}\theta$$ A1

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a}\pi y^2\mathrm{d}x=\pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{a\cos\theta}{a^2(1-\sin^2\theta)}\mathrm{d}\theta$$

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a}\pi y^2\mathrm{d}x=\pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{a\cos\theta}{a^2\cos^2\theta}\mathrm{d}\theta$$ A1

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a}\pi y^2\mathrm{d}x=\pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{a\cos\theta}\mathrm{d}\theta$$ M1

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a}\pi y^2\mathrm{d}x=\frac{\pi}{a}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec\theta\mathrm{d}\theta$$

$$\therefore\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a}\pi y^2\mathrm{d}x=\frac{\pi}{a}\Big[\ln\left|\sec\theta+\tan\theta\right|\Big]_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi}$$ A1

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a}\pi y^2\mathrm{d}x=\frac{\pi}{a}\left(\begin{array}{l}\ln\left|\sec\frac{1}{4}\pi+\tan\frac{1}{4}\pi\right|\\-\ln\left|\sec\left(-\frac{1}{4}\pi\right)+\tan\left(-\frac{1}{4}\pi\right)\right|\end{array}\right)$$

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a}\pi y^2\mathrm{d}x=\frac{\pi}{a}(\ln(\sqrt{2}+1)-\ln(\sqrt{2}-1))$$ A2

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a}\pi y^2\mathrm{d}x=\frac{\pi}{a}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}$$ A1

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a} \pi y^2 \mathrm{d}x = \frac{\pi}{a} \ln \frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}$$ M1

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a} \pi y^2 \mathrm{d}x = \frac{\pi}{a} \ln \frac{2+2\sqrt{2}+1}{2-1}$$ A1

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a} \pi y^2 \mathrm{d}x = \frac{\pi}{a} \ln(3+2\sqrt{2})$$ AG

[12]

3. (a) If $f(a)=f(b)$,

$2\cdot 3^a = 2\cdot 3^b$	M1
$3^a = 3^b$	
$a=b$	R1
Thus, f is a one-to-one function.	AG

[2]

(b)

$y=2\cdot 3^x$	
$\Rightarrow x=2\cdot 3^y$	(M1) for swapping variables
$\dfrac{x}{2}=3^y$	
$y=\log_3\dfrac{x}{2}$	
$\therefore f^{-1}(x)=\log_3\dfrac{x}{2}$	A1

[2]

(c)

$f'(x)=2\cdot 3^x\ln 3$	(A1) for correct approach
$f'(a)=\dfrac{2\cdot 3^a-0}{a-0}$	(M1) for setting equation
$\therefore 2\cdot 3^a\ln 3=\dfrac{2\cdot 3^a}{a}$	(A1) for substitution
$\ln 3=\dfrac{1}{a}$	
$a=\dfrac{1}{\ln 3}$	A1
The equation of L:	
$y-0=\dfrac{2\cdot 3^{\frac{1}{\ln 3}}}{\frac{1}{\ln 3}}(x-0)$	A1
$y=\left(2\ln 3\cdot 3^{\frac{1}{\ln 3}}\right)x$	A1

[6]

(d) The volume of the solid generated

$$= \int_0^{\frac{1}{\ln 3}} \pi (2\cdot 3^x)^2 \mathrm{d}x - \int_0^{\frac{1}{\ln 3}} \pi \left(\left(2\ln 3 \cdot 3^{\frac{1}{\ln 3}} \right) x \right)^2 \mathrm{d}x$$ M1A1

$$= \pi \int_0^{\frac{1}{\ln 3}} \left(4\cdot 9^x - 4(\ln 3)^2 \cdot 3^{\frac{2}{\ln 3}} x^2 \right) \mathrm{d}x$$ A1

$$= \pi \left[\frac{4\cdot 9^x}{\ln 9} - \frac{4}{3}(\ln 3)^2 \cdot 3^{\frac{2}{\ln 3}} x^3 \right]_0^{\frac{1}{\ln 3}}$$ A1

$$= \pi \left(\frac{4\cdot 9^{\frac{1}{\ln 3}}}{\ln 9} - \frac{4}{3}(\ln 3)^2 \cdot 3^{\frac{2}{\ln 3}} \left(\frac{1}{\ln 3} \right)^3 - \left(\frac{4\cdot 9^0}{\ln 9} - 0 \right) \right)$$

$$= \pi \left(\frac{4\cdot 9^{\frac{1}{\ln 3}}}{\ln 9} - \frac{4\cdot 3^{\frac{2}{\ln 3}}}{3\ln 3} - \frac{4}{\ln 9} \right)$$ M1

$$= \pi \left(\frac{4\cdot 9^{\frac{1}{\ln 3}}}{\ln 9} - \frac{4\cdot 9^{\frac{1}{\ln 3}}}{3\ln 3} - \frac{4}{\ln 9} \right)$$

$$= \pi \left(\frac{4\cdot 9^{\frac{1}{\ln 3}}}{2\ln 3} - \frac{4\cdot 9^{\frac{1}{\ln 3}}}{3\ln 3} - \frac{4}{2\ln 3} \right)$$ A1

$$= \pi \left(\frac{12\cdot 9^{\frac{1}{\ln 3}}}{6\ln 3} - \frac{8\cdot 9^{\frac{1}{\ln 3}}}{6\ln 3} - \frac{12}{6\ln 3} \right)$$

$$= \pi \left(\frac{12\cdot 9^{\frac{1}{\ln 3}} - 8\cdot 9^{\frac{1}{\ln 3}} - 12}{6\ln 3} \right)$$ M1

$$= \pi \left(\frac{4\cdot 9^{\frac{1}{\ln 3}} - 12}{6\ln 3} \right)$$ M1

$$= \pi \left(\frac{4\left(9^{\frac{1}{\ln 3}} - 3 \right)}{6\ln 3} \right)$$

$$= \frac{2\pi \left(9^{\frac{1}{\ln 3}} - 3 \right)}{3\ln 3}$$ AG

[8]

4. (a) If $f(a) = f(b)$,

$k\log_2 a = k\log_2 b$ M1

$\log_2 a = \log_2 b$

$a = b$ A1

Thus, f is a one-to-one function. AG

[2]

(b) $y = k\log_2 x$

$\Rightarrow x = k\log_2 y$ (M1) for swapping variables

$\frac{x}{k} = \log_2 y$

$y = 2^{\frac{x}{k}}$

$\therefore f^{-1}(x) = 2^{\frac{x}{k}}$ A1

[2]

(c) $\frac{1}{32} = 2^{-\frac{5}{k}}$ (M1) for setting equation

$2^{-5} = 2^{-\frac{5}{k}}$

$-5 = -\frac{5}{k}$

$k = 1$ A1

[2]

(d) $0 = \log_2 x$

$x = 1$ (A1) for correct value

The area of R

$= \int_1^{e^2} \log_2 x \mathrm{d}x$ (A1) for correct approach

$= \int_1^{e^2} \frac{\ln x}{\ln 2} \mathrm{d}x$ A1

$= \frac{1}{\ln 2} \int_1^{e^2} \ln x \mathrm{d}x$

$= \frac{1}{\ln 2}\left(\left[x \ln x \right]_1^{e^2} - \int_1^{e^2} x \mathrm{d}(\ln x) \right)$ (A2) for correct working

$= \frac{1}{\ln 2}\left(\left[x \ln x \right]_1^{e^2} - \int_1^{e^2} x \cdot \frac{1}{x} \mathrm{d}x \right)$

$= \frac{1}{\ln 2}\left(\left[x \ln x \right]_1^{e^2} - \int_1^{e^2} \mathrm{d}x \right)$ (M1) for valid approach

$= \frac{1}{\ln 2}\left(\left[x \ln x \right]_1^{e^2} - \left[x \right]_1^{e^2} \right)$ A1

$= \frac{1}{\ln 2}\left[x \ln x - x \right]_1^{e^2}$

$= \frac{1}{\ln 2}\left[(e^2 \ln e^2 - e^2) - (1 \ln 1 - 1) \right]_1^{e^2}$ A1

$= \frac{1}{\ln 2}(2e^2 - e^2 + 1)$

$= \frac{e^2 + 1}{\ln 2}$ A1

[9]

(e) $\int_0^b 2^x \mathrm{d}x = \frac{e^2 + 1}{\ln 2}$ M1A1

$\left[\frac{2^x}{\ln 2} \right]_0^b = \frac{e^2 + 1}{\ln 2}$ A1

$\frac{2^b}{\ln 2} - \frac{2^0}{\ln 2} = \frac{e^2 + 1}{\ln 2}$

$2^b - 1 = e^2 + 1$ A1

$2^b = e^2 + 2$

$b = \log_2(e^2 + 2)$ AG

[4]

Exercise 58

1. (a) $\log_8 y^3 + \log_2 x = \log_8 16\sqrt{2}$

$\log_8 y^3 + \dfrac{\log_8 x}{\log_8 2} = \log_8 16\sqrt{2}$ (A1) for correct formula

$\log_8 y^3 + \dfrac{\log_8 x}{\frac{1}{3}} = \log_8 16\sqrt{2}$ (A1) for correct value

$\log_8 y^3 + 3\log_8 x = \log_8 16\sqrt{2}$ (M1) for valid approach

$\log_8 y^3 + \log_8 x^3 = \log_8 16\sqrt{2}$ (A1) for correct approach

$\log_8 y^3 x^3 = \log_8 16\sqrt{2}$ (A1) for correct formula

$\therefore y^3 x^3 = 16\sqrt{2}$ M1

$yx = 2\sqrt{2}$

$x = 2\sqrt{2}y^{-1}$ A1

[7]

(b) $\int_1^c 2\sqrt{2}y^{-1}\mathrm{d}y = 2\sqrt{2}$ M1A1

$\left[2\sqrt{2}\ln y\right]_1^c = 2\sqrt{2}$ A1

$2\sqrt{2}\ln c - 2\sqrt{2}\ln 1 = 2\sqrt{2}$

$2\sqrt{2}\ln c = 2\sqrt{2}$ A1

$\ln c = 1$ M1

$c = e$ AG

[5]

(c) $\int_\alpha^{2\alpha}(f(y) - g(y))\mathrm{d}y = \int_\alpha^{2\alpha}(f(y) - (-f(y)))\mathrm{d}y$ A1

$\int_\alpha^{2\alpha}(f(y) - g(y))\mathrm{d}y = \int_\alpha^{2\alpha} 2f(y)\mathrm{d}y$

$\int_\alpha^{2\alpha}(f(y) - g(y))\mathrm{d}y = \int_\alpha^{2\alpha} 2(2\sqrt{2}y^{-1})\mathrm{d}y$ M1

$\int_\alpha^{2\alpha}(f(y) - g(y))\mathrm{d}y = \left[4\sqrt{2}\ln y\right]_\alpha^{2\alpha}$ A1

$\int_\alpha^{2\alpha}(f(y) - g(y))\mathrm{d}y = 4\sqrt{2}\ln 2\alpha - 4\sqrt{2}\ln\alpha$

$\int_\alpha^{2\alpha}(f(y) - g(y))\mathrm{d}y = 4\sqrt{2}\ln\dfrac{2\alpha}{\alpha}$ A1

$\int_\alpha^{2\alpha}(f(y) - g(y))\mathrm{d}y = 4\sqrt{2}\ln 2$ AG

[4]

(d) $\int_{125}^{8000}(f(y)-g(y))\mathrm{d}y$

$=\int_{125}^{250}(f(y)-g(y))\mathrm{d}y+\int_{250}^{500}(f(y)-g(y))\mathrm{d}y$

$+\int_{500}^{1000}(f(y)-g(y))\mathrm{d}y+\int_{1000}^{2000}(f(y)-g(y))\mathrm{d}y$ (A1) for correct approach

$+\int_{2000}^{4000}(f(y)-g(y))\mathrm{d}y+\int_{4000}^{8000}(f(y)-g(y))\mathrm{d}y$

$=\int_{125}^{2(125)}(f(y)-g(y))\mathrm{d}y+\int_{250}^{2(250)}(f(y)-g(y))\mathrm{d}y$

$+\int_{500}^{2(500)}(f(y)-g(y))\mathrm{d}y+\int_{1000}^{2(1000)}(f(y)-g(y))\mathrm{d}y$ (M1) for valid approach

$+\int_{2000}^{2(2000)}(f(y)-g(y))\mathrm{d}y+\int_{4000}^{2(4000)}(f(y)-g(y))\mathrm{d}y$

$=6(4\sqrt{2}\ln 2)$ A1

$=24\sqrt{2}\ln 2$ A1

[4]

2. (a)

$f(x) = g(x)$	
$4e^{2x} = 4e^{x+2}$	(M1) for setting equation
$e^{2x} = e^{x+2}$	
$e^{x} = e^{2}$	(A1) for correct approach
$x = 2$	(A1) for correct value
$f(2) = 4e^{2(2)}$	(M1) for substitution
$f(2) = 4e^{4}$	
Thus, the required coordinates are $(2, 4e^4)$.	A1

[5]

(b)

The area of R	
$= \int_0^2 (g(x) - f(x))\mathrm{d}x$	A1
$= \int_0^2 (4e^{x+2} - 4e^{2x})\mathrm{d}x$	
$= \left[4e^{x+2} - \frac{4}{2}e^{2x} \right]_0^2$	A1
$= \left[4e^{x+2} - 2e^{2x} \right]_0^2$	
$= (4e^{2+2} - 2e^{2(2)}) - (4e^{0+2} - 2e^{2(0)})$	M1
$= (4e^4 - 2e^4) - (4e^2 - 2)$	
$= 2e^4 - 4e^2 + 2$	M1
$= 2((e^2)^2 - 2e^2 + 1)$	A1
$= 2(e^2 - 1)^2$	A1
$= 2((e+1)(e-1))^2$	M1
$= 2(e+1)^2(e-1)^2$	AG

[7]

(c) $f'(x) = 4(e^{2x})(2)$

$f'(x) = 8e^{2x}$

$f''(x) = 8(e^{2x})(2)$

$f''(x) = 16e^{2x}$ A1

$g'(x) = 4e^{x+2}$

$g''(x) = 4e^{x+2}$ A1

$f''(x) = g''(x)$

$\therefore 16e^{2x} = 4e^{x+2}$ M1

$4e^{2x} = e^{x+2}$

$4e^{x} = e^{2}$

$e^{x} = \dfrac{e^2}{4}$ A1

$x = \ln \dfrac{e^2}{4}$ A1

$x = \ln e^2 - \ln 4$

$\therefore a = 2 - \ln 4$ AG

[5]

(d) $\text{PQ} = \left| f(2-\ln 4) - g(2-\ln 4) \right|$ (M1) for valid approach

$\text{PQ} = \left| 4e^{2(2-\ln 4)} - 4e^{2-\ln 4+2} \right|$

$\text{PQ} = \left| 4e^{4-2\ln 4} - 4e^{4-\ln 4} \right|$

$\text{PQ} = \left| 4e^{4-2\ln 4}(1 - e^{\ln 4}) \right|$ (A1) for correct approach

$\text{PQ} = \left| \dfrac{4e^4}{e^{\ln 16}}(1 - e^{\ln 4}) \right|$

$\text{PQ} = \left| \dfrac{4e^4}{16}(1-4) \right|$ (A1) for correct approach

$\text{PQ} = \left| \dfrac{e^4}{4}(-3) \right|$

$\text{PQ} = \dfrac{3}{4}e^4$ A1

[4]

3. (a) (i) $f(x) = g(x)$

$\therefore \sin \pi x = \sin 2\pi x$ (M1) for setting equation

$\sin \pi x = 2\sin \pi x \cos \pi x$ (A1) for substitution

$\sin \pi x - 2\sin \pi x \cos \pi x = 0$

$\sin \pi x(1 - 2\cos \pi x) = 0$ (A1) for factorization

$\sin \pi x = 0$ or $\cos \pi x = \dfrac{1}{2}$

$\pi x = 0$, $\pi x = \pi$ or $\pi x = \dfrac{\pi}{3}$ A1

$x = 0$ (*Rejected*), $x = 1$ (*Rejected*) or $x = \dfrac{1}{3}$

$\therefore r = \dfrac{1}{3}$ A1

(ii) The area of the region

$= \int_{\frac{1}{3}}^{1} (f(x) - g(x))\mathrm{d}x$ A1

$= \int_{\frac{1}{3}}^{1} (\sin \pi x - \sin 2\pi x)\mathrm{d}x$

$= \left[-\dfrac{1}{\pi}\cos \pi x + \dfrac{1}{2\pi}\cos 2\pi x \right]_{\frac{1}{3}}^{1}$ A1

$= \left(-\dfrac{1}{\pi}\cos \pi(1) + \dfrac{1}{2\pi}\cos 2\pi(1) \right)$
$- \left(-\dfrac{1}{\pi}\cos \pi\left(\dfrac{1}{3}\right) + \dfrac{1}{2\pi}\cos 2\pi\left(\dfrac{1}{3}\right) \right)$ M1

$= \left(-\dfrac{1}{\pi}(-1) + \dfrac{1}{2\pi}(1) \right)$
$- \left(-\dfrac{1}{\pi}\left(\dfrac{1}{2}\right) + \dfrac{1}{2\pi}\left(-\dfrac{1}{2}\right) \right)$ A1

$= \dfrac{1}{\pi} + \dfrac{1}{2\pi} + \dfrac{1}{2\pi} + \dfrac{1}{4\pi}$ M1

$= \dfrac{4}{4\pi} + \dfrac{2}{4\pi} + \dfrac{2}{4\pi} + \dfrac{1}{4\pi}$

$= \dfrac{9}{4\pi}$ AG

[10]

(b) The coordinates of Q are $\left(\frac{1}{4}, 1\right)$. (A1) for correct values

$\therefore a\sin\pi\left(\frac{1}{4}\right)=1$ (M1) for substitution

$\frac{\sqrt{2}}{2}a=1$ A1

$a=\sqrt{2}$ A1

[4]

(c) $f(x)>g(x)$

$\therefore a\sin\pi x>\sin 2\pi x$ (M1) for setting inequality

$a\sin\pi x>2\sin\pi x\cos\pi x$

$a>2\cos\pi x$ (A1) for correct approach

$0<x<1$

$0<\pi x<\pi$

$-1<\cos\pi x<1$ (A1) for correct values

$\therefore 2\cos\pi x<2$

Thus, the least possible value of a is 2. A1

[4]

4. (a) (i) $f(x)=g(x)$

$\therefore \cos 2\pi y=\cos \pi y$ M1

$2\cos^2 \pi y-1=\cos \pi y$ A1

$2\cos^2 \pi y-\cos \pi y-1=0$ A1

$(2\cos \pi y+1)(\cos \pi y-1)=0$ A1

$\cos \pi y=-\dfrac{1}{2}$ or $\cos \pi y=1$

$\pi y=\dfrac{2\pi}{3}$ or $\pi y=0$ A1

$y=\dfrac{2}{3}$ or $y=0$ (*Rejected*)

$\therefore r=\dfrac{2}{3}$ AG

(ii) The area of the region

$=\int_0^{\frac{2}{3}}(g(y)-f(y))\mathrm{d}y$ A1

$=\int_0^{\frac{2}{3}}(\cos \pi y-\cos 2\pi y)\mathrm{d}y$

$=\left[\dfrac{1}{\pi}\sin \pi y-\dfrac{1}{2\pi}\sin 2\pi y\right]_0^{\frac{2}{3}}$ A1

$=\left(\dfrac{1}{\pi}\sin \pi\left(\dfrac{2}{3}\right)-\dfrac{1}{2\pi}\sin 2\pi\left(\dfrac{2}{3}\right)\right)$

$-\left(\dfrac{1}{\pi}\sin \pi(0)-\dfrac{1}{2\pi}\sin 2\pi(0)\right)$ M1

$=\left(\dfrac{1}{\pi}\left(\dfrac{\sqrt{3}}{2}\right)-\dfrac{1}{2\pi}\left(-\dfrac{\sqrt{3}}{2}\right)\right)-0$ A1

$=\dfrac{\sqrt{3}}{2\pi}+\dfrac{\sqrt{3}}{4\pi}$ M1

$=\dfrac{2\sqrt{3}}{4\pi}+\dfrac{\sqrt{3}}{4\pi}$

$=\dfrac{3\sqrt{3}}{4\pi}$ AG

[10]

(b) $a\cos 2\pi\left(\frac{1}{6}\right)=\frac{\sqrt{3}}{2}$ (M1) for substitution

$\frac{1}{2}a=\frac{\sqrt{3}}{2}$ A1

$a=\sqrt{3}$ A1

[3]

(c) $f(x)=g(x)$

$\therefore a\cos 2\pi y=\cos\pi y$ M1

$a(2\cos^2\pi y-1)=\cos\pi y$ A1

$2a\cos^2\pi y-a=\cos\pi y$

$2a\cos^2\pi y-\cos\pi y-a=0$ A1

$\cos\pi y=\frac{-(-1)\pm\sqrt{(-1)^2-4(2a)(-a)}}{2(2a)}$ M1A1

$\cos\pi y=\frac{1\pm\sqrt{1+8a^2}}{4a}$

$\cos\pi y=\frac{1+\sqrt{1+8a^2}}{4a}$ or

$\cos\pi y=\frac{1-\sqrt{1+8a^2}}{4a}$ (*Rejected*) A1

$\pi y=\arccos\left(\frac{1+\sqrt{1+8a^2}}{4a}\right)$ M1

$\therefore r=\frac{1}{\pi}\arccos\left(\frac{1+\sqrt{1+8a^2}}{4a}\right)$ AG

[7]

Exercise 59

1. (a) $g(x)=\ln(x-7)^3$

$g(x)=3\ln(x-7)$ (M1) for valid approach

$g(x)=3f(x-7)$ (M1) for valid approach

$\therefore p=7,\ q=3$ A2

[4]

(b) The volume generated

$=\int_{3\pi}^{10}\pi(g(x))^2\,\mathrm{d}x$

$=\int_{3\pi}^{10}\pi(\ln(x-7)^3)^2\,\mathrm{d}x$ (A1) for correct approach

$=16.19380939$

$=16.2$ A1

[2]

2. (a) $g(x)=2^x$

$g(x)=8^{\frac{1}{3}x}-5+5$ (M1) for valid approach

$g(x)=f\left(\frac{1}{3}x\right)+5$ (M1) for valid approach

$\therefore p=\frac{1}{3},\ q=5$ A2

[4]

(b) The volume generated

$=\int_{0}^{\pi}\pi(g(x))^2\,\mathrm{d}x$

$=\int_{0}^{\pi}\pi(2^x)^2\,\mathrm{d}x$ (A1) for correct approach

$=174.2244533$

$=174$ A1

[2]

3. $y = \sec \pi(0)$

$y = 1$ (A1) for correct value

$y = \sec \pi x$

$y = \dfrac{1}{\cos \pi x}$ (M1) for valid approach

$\cos \pi x = \dfrac{1}{y}$

$\pi x = \arccos \dfrac{1}{y}$ (M1) for valid approach

$x = \dfrac{1}{\pi} \arccos \dfrac{1}{y}$ A1

The volume generated

$= \int_1^e \pi \left(\dfrac{1}{\pi} \arccos \dfrac{1}{y} \right)^2 \mathrm{d}y$ (A1) for correct approach

$= 0.5007292734$

$= 0.501$ A1

[6]

4. When $x = 0$,

$y = e^{4(0)}$

$y = 1$ (A1) for correct value

$y = e^{4x}$

$\ln y = 4x$ (M1) for valid approach

$x = \dfrac{1}{4} \ln y$ A1

The volume generated

$= \int_1^e \pi \left(\dfrac{1}{4} \ln y \right)^2 \mathrm{d}y$ (A1) for correct approach

$= 0.1410343072$

$= 0.141$ A1

[5]

Exercise 60

1. $0 = e^{2x} - 1$

$1 = e^{2x}$

$2x = 0$

$x = 0$ (A1) for correct value

$x + y - e^6 - 2 = 0$

$y = -x + e^6 + 2$ (A1) for correct approach

$e^{2x} - 1 = -x + e^6 + 2$ (M1) for setting equation

$e^{2x} - 1 = e^6 - (x - 2)$

$2x = 6$

$x = 3$ (A1) for correct value

$x + 0 - e^6 - 2 = 0$

$x = e^6 + 2$ (A1) for correct value

The area of R

$= \int_0^3 (e^{2x} - 1)\,dx + \int_3^{e^6+2} (-x + e^6 + 2)\,dx$ A1

$= 81172.68131$

$= 81200$ A1

[7]

2. $0 = 2\ln x$

$\ln x = 0$

$x = 1$ (A1) for correct value

$4x + e^2 y - 8e^2 = 0$

$e^2 y = 8e^2 - 4x$

$y = 8 - \frac{4}{e^2}x$ (A1) for correct approach

$2\ln x = 8 - \frac{4}{e^2}x$ (M1) for setting equation

$2\ln x + \frac{4}{e^2}x - 8 = 0$

By considering the graph of $y = 2\ln x + \frac{4}{e^2}x - 8$,

$x = 7.3890561$. (A1) for correct value

$4x + e^2(0) - 8e^2 = 0$

$4x = 8e^2$

$x = 2e^2$ (A1) for correct value

The volume of the solid

$= \int_1^{7.3890561} \pi(2\ln x)^2 \, dx + \int_{7.3890561}^{2e^2} \pi\left(8 - \frac{4}{e^2}x\right)^2 dx$ A1

$= 284.3793169$

$= 284$ A1

[7]

3. $y = 5^{-x}$

$\log_5 y = -x$

$x = -\log_5 y$ (A1) for correct approach

$y = 5^{-0}$

$y = 1$ (A1) for correct value

$y = 25x + 75$

$y - 75 = 25x$

$x = \frac{1}{25}y - 3$ (A1) for correct approach

$-\log_5 y = \frac{1}{25}y - 3$ (M1) for setting equation

$\frac{1}{25}y - 3 + \log_5 y = 0$

By considering the graph of $x = \frac{1}{25}y - 3 + \log_5 y$,

$y = 25$. (A1) for correct value

$y = 25(0) + 75$

$y = 75$ (A1) for correct value

The area of R

$= -\int_1^{25}(-\log_5 y)\mathrm{d}y - \int_{25}^{75}\left(\frac{1}{25}y - 3\right)\mathrm{d}y$ A1

$= 85.08796157$

$= 85.1$ A1

[8]

4. $y = \frac{1}{2}x + 1$

$2y = x + 2$

$x = 2y - 2$ (A1) for correct approach

$y = \frac{1}{2}(0) + 1$

$y = 1$ (A1) for correct value

$y = (x-4)^2 + 3$

$y - 3 = (x-4)^2$

$x - 4 = \sqrt{y-3}$ or $x - 4 = -\sqrt{y-3}$

$x = 4 + \sqrt{y-3}$ (*Rejected*) or $x = 4 - \sqrt{y-3}$ (A1) for correct approach

$2y - 2 = 4 - \sqrt{y-3}$ (M1) for setting equation

$2y - 6 = -\sqrt{y-3}$

$(2y-6)^2 = y - 3$

$4y^2 - 24y + 36 = y - 3$

$4y^2 - 25y + 39 = 0$ (A1) for correct approach

$(y-3)(4y-13) = 0$

$y = 3$ or $y = \frac{13}{4}$ (*Rejected*) (A1) for correct value

$y = (0-4)^2 + 3$

$y = 19$ (A1) for correct value

The volume of the solid

$= \int_1^3 \pi(2y-2)^2 \, dy + \int_3^{19} \pi(4 - \sqrt{y-3})^2 \, dy$ A1

$= 167.551608$

$= 168$ A1

[9]

Exercise 61

1. (a) The initial velocity

$= v(0)$ (M1) for valid approach

$= (12 - 1.5(0))^3$

$= 1728 \text{ ms}^{-1}$ A1

[2]

(b) $v(t) = 3.375$ (M1) for setting equation

$(12 - 1.5t)^3 = 3.375$

$12 - 1.5t = 1.5$ (A1) for correct approach

$-1.5t = -10.5$

$t = 7$ A1

[3]

(c) The total distance travelled

$= \int_7^{11} |v(t)| \, dt$ (M1) for valid approach

$= \int_7^{11} |(12 - 1.5t)^3| \, dt$ (A1) for substitution

$= 69.1875 \text{ m}$ A1

[3]

(d) $a(t) = v'(t)$ M1

$a(t) = 3(12 - 1.5t)^2(-1.5)$ A2

$a(t) = -4.5(12 - 1.5t)^2$ AG

[3]

(e) $v(t) < a(t)$

$(12 - 1.5t)^3 < -4.5(12 - 1.5t)^2$

$(12 - 1.5t)^3 + 4.5(12 - 1.5t)^2 < 0$ (M1) for setting inequality

By considering the graph of

$y = (12 - 1.5t)^3 + 4.5(12 - 1.5t)^2$, $11 < t \le 16$. A1

[2]

(f) $a(t) = -4.5(12-1.5t)^2$

$$\frac{dv}{dt} = -4.5(12-1.5t)^2$$

$$\frac{dv}{ds}\cdot\frac{ds}{dt} = -4.5(12-1.5t)^2$$ A1

$$\frac{dv}{ds}\cdot(12-1.5t)^3 = -4.5(12-1.5t)^2$$ M1

$$\frac{dv}{ds} = -\frac{4.5(12-1.5t)^2}{(12-1.5t)^3}$$ A1

$$\frac{dv}{ds} = -\frac{4.5}{12-1.5t}\ \text{s}^{-1}$$ AG

[3]

2. (a) $v(t)=0.01$ (M1) for setting equation

$-1+\ln|2+\cos t|=0.01$

$\ln|2+\cos t|=1.01$

$2+\cos t=e^{1.01}$ (A1) for correct approach

$\cos t=e^{1.01}-2$

$t=0.7293600469$

$t=0.729$ A1

[3]

(b) $|v(t)|<0.05$

$|-1+\ln|2+\cos t||<0.05$

$|-1+\ln|2+\cos t||-0.05<0$ (M1) for setting inequality

By considering the graph of

$y=|-1+\ln|2+\cos t||-0.05$,

$0.540112<t<0.945041$. (M1) for valid approach

$\therefore 0.540<t<0.945$ A1

[3]

(c) The total distance travelled

$=\int_0^1 |v(t)|\,dt$ (M1) for valid approach

$=\int_0^1 |-1+\ln|2+\cos t||\,dt$ (A1) for substitution

$=0.0580494843\text{ m}$

$=0.0580\text{ m}$ A1

[3]

(d) $a(t) = v'(t)$ (M1) for valid approach

$a(t) = 0 + \left(\frac{1}{2+\cos t}\right)(-\sin t)$ (A1) for chain rule

$a(t) = -\frac{\sin t}{2+\cos t}$ A1

$v(t) = 0$

$-1 + \ln|2+\cos t| = 0$

By considering the graph of $y = -1 + \ln|2+\cos t|$,

$t = 0.7694667$. (A1) for correct value

The required acceleration

$= a(0.7694667)$

$= -\frac{\sin 0.7694667}{2+\cos 0.7694667}$ (M1) for substitution

$= -0.2559529603$

$= -0.256 \text{ ms}^{-2}$ A1

[6]

(e) $v(t) \cdot a(t) < \frac{1}{2}$

$(-1+\ln|2+\cos t|)\left(-\frac{\sin t}{2+\cos t}\right) < \frac{1}{2}$

$(-1+\ln|2+\cos t|)\left(-\frac{\sin t}{2+\cos t}\right) - \frac{1}{2} < 0$

$(-1+\ln|2+\cos t|)\left(\frac{\sin t}{2+\cos t}\right) + \frac{1}{2} > 0$ (M1) for setting inequality

By considering the graph of

$y = (-1+\ln|2+\cos t|)\left(\frac{\sin t}{2+\cos t}\right) + \frac{1}{2}$,

$t > 0.9118124$. (M1) for valid approach

$\therefore 0.912 < t \le 1$ A1

[3]

3. (a) When $0 \le t \le 1$,

$s(t) = \int -\frac{1}{10} t \mathrm{d}t$ (M1) for valid approach

$s(t) = -\frac{1}{20} t^2 + C$ (A1) for correct value

$s(0) = 1$

$\therefore -\frac{1}{20}(0)^2 + C = 1$

$C = 1$

$s(1) = -\frac{1}{20}(1)^2 + 1$

$s(1) = \frac{19}{20}$ (A1) for correct value

When $1 < t \le 3$,

$s(t) = \int \left(\frac{1}{10} t^2 - \frac{1}{5} t \right) \mathrm{d}t$ (M1) for valid approach

$s(t) = \frac{1}{30} t^3 - \frac{1}{10} t^2 + D$ (A1) for correct value

$s(1) = \frac{19}{20}$

$\therefore \frac{1}{30}(1)^3 - \frac{1}{10}(1)^2 + D = \frac{19}{20}$

$D = \frac{61}{60}$ (A1) for correct value

$s(3) = \frac{1}{30}(3)^3 - \frac{1}{10}(3)^2 + \frac{61}{60}$

$s(3) = \frac{61}{60}$

$$\therefore s(t) = \begin{cases} -\frac{1}{20} t^2 + 1 & 0 \le t \le 1 \\ \frac{1}{30} t^3 - \frac{1}{10} t^2 + \frac{61}{60} & 1 < t \le 3 \\ \frac{61}{60} & t > 3 \end{cases}$$ A1

[7]

(b) $s(t)=1$

$\therefore \frac{1}{30}t^3-\frac{1}{10}t^2+\frac{61}{60}=1$

$\frac{1}{30}t^3-\frac{1}{10}t^2+\frac{1}{60}=0$ (M1) for setting equation

By considering the graph of $y=\frac{1}{30}t^3-\frac{1}{10}t^2+\frac{1}{60}$,

$t=2.9422419$. (M1) for valid approach

$\therefore t=0$ or $t=2.94$ A1

[3]

(c) $a(t)=\begin{cases} -\frac{1}{10}(1) & 0\le t\le 1 \\ \frac{1}{10}(2t)-\frac{1}{5}(1) & 1<t\le 3 \\ 0 & \text{otherwise} \end{cases}$ (M1) for valid approach

$a(t)=\begin{cases} -\frac{1}{10} & 0\le t\le 1 \\ \frac{1}{5}t-\frac{1}{5} & 1<t\le 3 \\ 0 & \text{otherwise} \end{cases}$ (A1) for correct values

$a(t)<\frac{3}{10}$

$\frac{1}{5}t-\frac{1}{5}<\frac{3}{10}$ (M1) for setting inequality

$\frac{1}{5}t<\frac{1}{2}$

$t<\frac{5}{2}$

$\therefore 0\le t<\frac{5}{2}$ A1

[4]

(d) $v = \frac{1}{10}t^2 - \frac{1}{5}t$

$10v = t^2 - 2t$

$10v + 1 = t^2 - 2t + 1$ A1

$10v + 1 = (t-1)^2$

$t - 1 = \sqrt{10v+1}$

$t = 1 + \sqrt{10v+1}$ A1

$\frac{dt}{dv} = 0 + \left(\frac{1}{2\sqrt{10v+1}}\right)(10)$ M1A1

$\therefore \frac{dt}{dv} = \frac{5}{\sqrt{10v+1}}$ for $1 < t \leq 3$. AG

[4]

4. (a) $v(t)=\int a(t)\mathrm{d}t$ M1

$v(t)=\int 38\ln 3\times 3^{-0.38t}\mathrm{d}t$

$v(t)=38\ln 3\left(\dfrac{3^{-0.38t}}{-0.38\ln 3}\right)+C$ A1

$v(t)=-100\times 3^{-0.38t}+C$ A1

$0=-100\times 3^{-0.38(0)}+C$ M1

$0=-100+C$

$C=100$

$s(t)=\int v(t)\mathrm{d}t$ M1

$s(t)=\int(-100\times 3^{-0.38t}+100)\mathrm{d}t$

$s(t)=-100\left(\dfrac{3^{-0.38t}}{-0.38\ln 3}\right)+100t+D$ A1

$s(t)=\dfrac{5000}{19\ln 3}\times 3^{-0.38t}+100t+D$ A1

$\dfrac{5000}{19\ln 3}=\dfrac{5000}{19\ln 3}\times 3^{-0.38(0)}+100(0)+D$ M1

$\dfrac{5000}{19\ln 3}=\dfrac{5000}{19\ln 3}+D$

$D=0$

$\therefore s(t)=\dfrac{5000}{19\ln 3}\times 3^{-0.38t}+100t$ A1

$s(t)=\dfrac{5000}{19\ln 3}\left(3^{-0.38t}+\dfrac{19\ln 3}{50}t\right)$

$s(t)=\dfrac{5000}{19\ln 3}(3^{-0.38t}+(0.38\ln 3)t)$ AG

[9]

(b) $s'(t)=a(t)$

$v(t)=a(t)$ (M1) for setting equation

$-100\times 3^{-0.38t}+100=38\ln 3\times 3^{-0.38t}$

$(38\ln 3+100)\times 3^{-0.38t}-100=0$

By considering the graph of

$y=(38\ln 3+100)\times 3^{-0.38t}-100$, $t=0.8356846$. (M1) for valid approach

$\therefore t=0.836$ A1

[3]

(c) The total distance travelled

$= \int_6^{10} |v(t)| \, dt$ (M1) for valid approach

$= \int_6^{10} |-100 \times 3^{-0.38t} + 100| \, dt$ (A1) for substitution

$= 384.1164218 \text{ m}$

$= 384 \text{ m}$ A1

[3]

(d) $v = -100 \times 3^{-0.38t} + 100$

$v = 100(1 - 3^{-0.38t})$

$0.01v = 1 - 3^{-0.38t}$

$3^{-0.38t} = 1 - 0.01v$ A1

$\frac{dv}{dt} = 38 \ln 3 \times 3^{-0.38t}$ A1

$\frac{dv}{dt} = 38 \ln 3(1 - 0.01v)$ M1

$\frac{dv}{dt} = 38 \ln 3\left(\frac{100 - v}{100}\right)$

$\therefore \frac{dt}{dv} = \frac{50}{19 \ln 3(100 - v)}$ AG

[3]