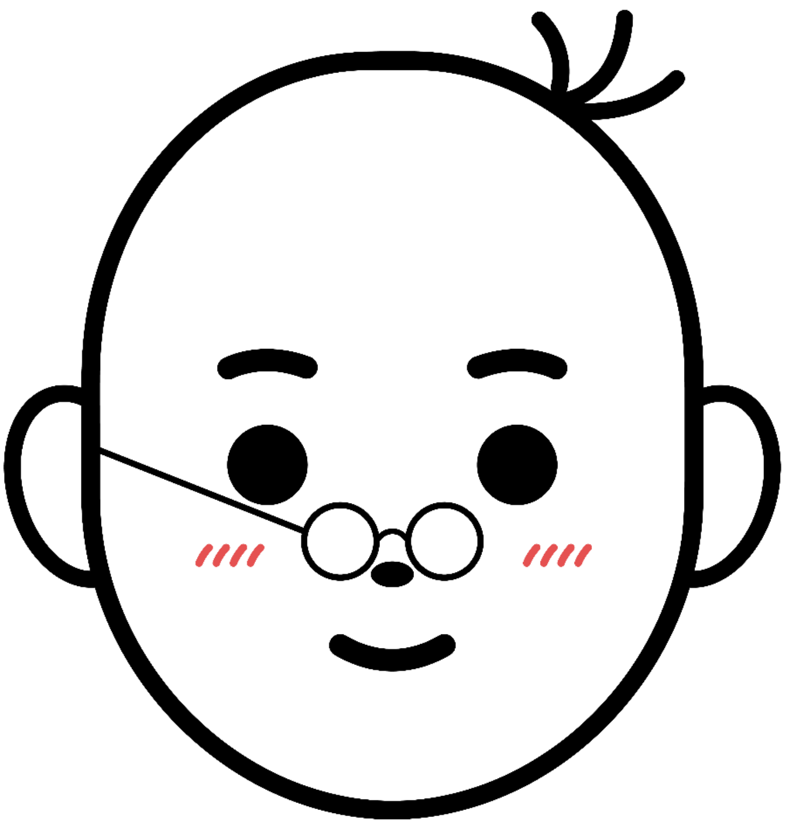
Your Intensive Notes
Analysis and Approaches
Higher Level
for IBDP Mathematics



Algebra

Topics Covered

1 Standard Form

Important Notes

Standard form: A number in the format $\pm a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$ (k is an integer)

Notes on GDC		
TEXAS TI-84 Plus CE	**TEXAS TI-Nspire CX**	**CASIO fx-CG50**
mode→SCI on the 2nd row to express any number in its standard form	Doc→Setting & Status →Document Settings… →Scientific on the Exponential Format row to express any number in its standard form	SHIFT MENU →Sci on the Display row →3 to express any number in its standard form

Example 1.1

A rectangle is 2376 cm long and 693 cm wide.

(a) Find the diagonal length of the rectangle, giving the answer in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$.

[2]

The required diagonal length

$= \sqrt{2376^2 + 693^2}$ Pythagoras' theorem (A1)

$= 2475\text{ cm}$

$= 2.475 \times 10^3\text{ cm}$ $a = 2.475$ & $k = 3$ (A1)

(b) Find the area of the rectangle, giving the answer correct to two significant figures and in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$.

[2]

The required area

$= (2376)(693)$ Base Length × Height (A1)

$= 1646568\text{ cm}^2$

$= 1600000\text{ cm}^2$ Round off to 2 sig. fig.

$= 1.6 \times 10^6\text{ cm}^2$ $a = 1.6$ & $k = 6$ (A1)

Exercise 1.1

The base length and the height of a right-angled triangle are 1107 mm and 4920 mm respectively.

(a) Find the length of the hypotenuse of the triangle, giving the answer in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$.

[2]

(b) Using (a), find the perimeter of the triangle, giving the answer correct to two significant figures and in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$.

[2]

(c) Find the area of the triangle, giving the answer correct to four significant figures and in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$.

[2]

2 Arithmetic Sequences

Important Notes

An arithmetic sequence is a sequence such that the next term is generated by adding or subtracting the same number from the previous term.

Properties of an arithmetic sequence $u_1, u_2, u_3, \cdots$:

1. u_1: First term
2. $d = u_2 - u_1 = u_n - u_{n-1}$: Common difference
3. $u_n = u_1 + (n-1)d$: General term (nth term)
4. $S_n = \frac{n}{2}[2u_1 + (n-1)d] = \frac{n}{2}[u_1 + u_n]$: Sum of the first n terms

$\sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + \cdots + u_{n-1} + u_n$: Summation sign

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[y=] to input the general term →[2nd] [window] to set the starting row →[2nd] [graph] to find the value of the term needed	Graph to input the general term to generate a table →[ctrl] [T] to generate a table →[menu] [2] [5] to set the starting row to find the value of the term needed	Table to input the general term to generate a table →[F5] to set the starting row →[F6] to find the value of the term needed

Example 1.2

Consider the arithmetic sequence $15, 21, 27, \cdots$.

(a) Write down d, the common difference of this sequence.

[1]

$d = 21 - 15$ — $d = u_2 - u_1$

$d = 6$ — (A1)

(b) Find u_8.

[2]

$u_8 = u_1 + (8-1)d$ — $u_n = u_1 + (n-1)d$

$u_8 = 15 + (8-1)(6)$ — Correct approach (A1)

$u_8 = 15 + 42$

$u_8 = 57$ — (A1)

(c) Find n such that $u_n = 75$.

[2]

$u_n = 75$ — Set up an equation

$\therefore 15 + (n-1)(6) = 75$ — Correct equation (A1)

$6(n-1) = 60$

$n - 1 = 10$

$n = 11$ — (A1)

(d) Find an expression of the sum of the first n terms of this sequence.

[3]

The sum of the first n terms

$= S_n$

$= \frac{n}{2}\left[2u_1 + (n-1)d\right]$ — $S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$ (M1)

$= \frac{n}{2}\left[2(15) + (n-1)(6)\right]$ — $u_1 = 15$ & $d = 6$ (A1)

$= \frac{n}{2}(30 + 6n - 6)$

$= \frac{n}{2}(6n + 24)$

$= 3n^2 + 12n$ — (A1)

(e) Hence, find the sum of the first ten terms of this sequence.

[2]

The required sum

$= S_{10}$

$= 3(10)^2 + 12(10)$ $n = 10$ (M1)

$= 300 + 120$

$= 420$ (A1)

Exercise 1.2

Consider the arithmetic sequence $\frac{39}{10}, \frac{38}{10}, \frac{37}{10}, \cdots$.

(a) Write down d, the common difference of this sequence.

[1]

(b) Find u_{12}.

[2]

(c) Find n such that $u_n = \frac{7}{10}$.

[2]

(d) Find an expression of the sum of the first n terms of this sequence.

[3]

(e) Hence, find the sum of the first ten terms of this sequence.

[2]

Example 1.3

Consider the arithmetic sequence with first term 70 and common difference -3.5. It is given that the r th term of the sequence is zero.

(a) Find r.

[2]

$u_r = 0$ — Set up an equation

$\therefore 70 + (r-1)(-3.5) = 0$ — Correct equation (A1)

$-3.5(r-1) = -70$

$r - 1 = 20$

$r = 21$ — (A1)

(b) Find the maximum value of the sum of the first n terms of this sequence.

[3]

The sum of the first n terms

$= S_n$

$= \frac{n}{2}\left[2u_1 + (n-1)d\right]$ — $S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$ (M1)

$= \frac{n}{2}\left[2(70) + (n-1)(-3.5)\right]$

$= \frac{n}{2}(140 - 3.5n + 3.5)$

$= \frac{n(143.5 - 3.5n)}{2}$

By considering the graph of $y = \frac{n(143.5 - 3.5n)}{2}$,

the coordinates of the maximum point are $(20.5, 735.4375)$, and the graph passes through $(20, 735)$ and $(21, 735)$. — GDC approach (M1)

Thus, the maximum value is 735. — (A1)

Exercise 1.3

Consider the arithmetic sequence with first term 120 and common difference -1.25. It is given that the m th term of the sequence is zero.

(a) Find m.

[2]

(b) Find the maximum value of the sum of the first n terms of this sequence.

[3]

3 Geometric Sequences

Important Notes

A geometric sequence is a sequence such that the next term is generated by multiplying or being divided by the same number from the previous term.

Properties of a geometric sequence $u_1, u_2, u_3, \cdots$:

1. u_1: First term
2. $r = u_2 \div u_1 = u_n \div u_{n-1}$: Common ratio
3. $u_n = u_1 \times r^{n-1}$: General term (nth term)
4. $S_n = \frac{u_1(1-r^n)}{1-r} = \frac{u_1(r^n - 1)}{r-1}$: Sum of the first n terms
5. $S_\infty = u_1 + u_2 + u_3 + \cdots + u_n + \cdots = \frac{u_1}{1-r}$: Sum of infinite number of terms (Sum to infinity), valid only when $-1 < r < 1$

Properties of compound interest:

1. PV: Present value
2. $r\%$: Nominal annual interest rate
3. k: Number of compounded periods in one year
4. n: Number of years
5. $FV = PV\left(1 + \frac{r\%}{k}\right)^{kn}$: Future value
6. $I = FV - PV$: Interest

Notes on GDC		
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Example 1.4

Consider the sequence $\ln x, a\ln x, \frac{1}{2}\ln x, \cdots$, $x>0$, $a\neq 0$, x, $a\in\mathbb{R}$.

(a) Consider the case where the sequence is arithmetic.

(i) Find a.

[2]

$a\ln x-\ln x=\frac{1}{2}\ln x-a\ln x$ $d=u_2-u_1=u_3-u_2$ (A1)

$a-1=\frac{1}{2}-a$

$2a=\frac{3}{2}$

$a=\frac{3}{4}$ (A1)

(ii) Hence, express the common difference in terms of $\ln x$.

[2]

The common difference

$=\frac{3}{4}\ln x-\ln x$ $d=u_2-u_1$ (A1)

$=-\frac{1}{4}\ln x$ (A1)

(b) Consider the case where the sequence is geometric.

(i) Find the possible values of a.

[3]

$a\ln x\div\ln x=\frac{1}{2}\ln x\div a\ln x$ $r=u_2\div u_1=u_3\div u_2$ (A1)

$a=\frac{1}{2a}$

$2a^2=1$

$a^2=\frac{1}{2}$

$a=\frac{1}{\sqrt{2}}$ or $a=-\frac{1}{\sqrt{2}}$ (A1)(A1)

(ii) Hence, find an expression of the sum of the first n terms of this sequence if $a > 0$, giving the answer in terms of $\ln x$ and 2^M, $M \in \mathbb{R}$.

[4]

The sum of the first n terms

$= S_n$

$= \frac{u_1(1-r^n)}{1-r}$ — $S_n = \frac{u_1(1-r^n)}{1-r}$ (M1)

$= \frac{(\ln x)\left(1-\left(\frac{1}{\sqrt{2}}\right)^n\right)}{1-\frac{1}{\sqrt{2}}}$ — $u_1 = \ln x$ & $r = \frac{1}{\sqrt{2}}$ (A1)

$= \frac{(\ln x)\left(1-\left(\frac{1}{2^{0.5}}\right)^n\right)}{1-\frac{1}{2^{0.5}}}$ — $\sqrt{2} = 2^{0.5}$ (M1)

$= \frac{(\ln x)(1-(2^{-0.5})^n)}{1-2^{-0.5}}$

$= \frac{1-2^{-0.5n}}{1-2^{-0.5}} \ln x$ — (A1)

It is given that $S_\infty = 2-\sqrt{2}$ when $a < 0$.

(iii) Find x.

[3]

$S_\infty = 2-\sqrt{2}$ — Set up an equation

$\therefore \frac{\ln x}{1-\left(-\frac{1}{\sqrt{2}}\right)} = 2-\sqrt{2}$ — Correct equation (A1)

$\ln x = (2-\sqrt{2})\left(1+\frac{1}{\sqrt{2}}\right)$

$\ln x = 2+\sqrt{2}-\sqrt{2}-1$

$\ln x = 1$ — Simplify the R.H.S. (A1)

$x = e^1$

$x = e$ — (A1)

Exercise 1.4

Consider the sequence $\log x, k\log x, \frac{1}{5}\log x, \cdots$, $x > 0$, $k \neq 0$, x, $k \in \mathbb{R}$.

(a) Consider the case where the sequence is arithmetic.

(i) Find k.

[2]

(ii) Hence, express the common difference in terms of $\log x$.

[2]

(b) Consider the case where the sequence is geometric.

(i) Find the possible values of k.

[3]

(ii) Hence, find an expression of the sum of the first n terms of this sequence if $k < 0$, giving the answer in terms of $\log x$ and 5^M, $M \in \mathbb{R}$.

[4]

It is given that $S_\infty = \frac{5+\sqrt{5}}{2}$ when $k > 0$.

(iii) Find x.

[3]

Example 1.5

Consider the geometric sequence $\frac{1}{243}, \frac{1}{81}, \frac{1}{27}, \cdots$.

(a) Write down r, the common ratio of this sequence.

[1]

$r = \frac{1}{81} \div \frac{1}{243}$ — $r = u_2 \div u_1$

$r = 3$ — (A1)

(b) Find u_{12}.

[2]

$u_{12} = u_1 \times r^{12-1}$ — $u_n = u_1 \times r^{n-1}$

$u_{12} = \frac{1}{243} \times 3^{12-1}$ — $u_1 = \frac{1}{243}$ & $r = 3$ (A1)

$u_{12} = 729$ — (A1)

(c) Find n such that $u_n = 81$.

[2]

$u_n = 81$ — Set up an equation

$\therefore \frac{1}{243} \times 3^{n-1} = 81$ — Correct equation (A1)

$\frac{1}{243} \times 3^{n-1} - 81 = 0$

By considering the graph of $y = \frac{1}{243} \times 3^{n-1} - 81$,

the horizontal intercept is 10.

$\therefore n = 10$ — (A1)

(d) Find an expression of the sum of the first n terms of this sequence. [3]

The sum of the first n terms

$= S_n$

$= \dfrac{u_1(1-r^n)}{1-r}$ $\quad$ $S_n = \dfrac{u_1(1-r^n)}{1-r}$ (M1)

$= \dfrac{\frac{1}{243}(1-3^n)}{1-3}$ $\quad$ $u_1 = \dfrac{1}{243}$ & $r = 3$ (A1)

$= -\dfrac{1}{486}(1-3^n)$ $\quad$ (A1)

(e) Hence, find the sum of the first twelve terms of this sequence. [2]

The required sum

$= S_{12}$

$= -\dfrac{1}{486}(1-3^{12})$ $\quad$ $n = 12$ (M1)

$= 1093.497942$

$= 1090$ $\quad$ (A1)

(f) Explain why the sum to infinity of this geometric sequence does not exist. [1]

The common ratio is 3 which is not between -1 and 1. (R1)

(g) Find the least value of n such that $S_n > 10000$. [3]

$S_n > 10000$ $\quad$ Set up an inequality

$\therefore -\dfrac{1}{486}(1-3^n) > 10000$ $\quad$ Correct inequality (A1)

$-\dfrac{1}{486}(1-3^n) - 10000 > 0$

By considering the graph of $y = -\dfrac{1}{486}(1-3^n) - 10000$, the graph is above the horizontal axis when $n > 14.014543$. $\quad$ GDC approach (M1)

$\therefore$ The least value of n is 15. (A1)

Exercise 1.5

Consider the geometric sequence $\frac{7}{12}, \frac{7}{16}, \frac{21}{48}, \cdots$.

(a) Write down r, the common ratio of this sequence.

[1]

(b) Find u_7.

[2]

(c) Find n such that $u_n = \frac{189}{1024}$.

[2]

(d) Find an expression of the sum of the first n terms of this sequence.

[3]

(e) Hence, find the sum of the first fifteen terms of this sequence.

[2]

(f) Explain why the sum to infinity of this geometric sequence exists.

[1]

(g) Find the greatest value of n such that $S_n < 2.3315$.

[3]

Example 1.6

On 1st January 2024, Judy invests $\$P$ in an account that pays a nominal annual interest rate of 6%, compounded **half-yearly**. The amount of money in her account at the end of each year follows a geometric sequence with common ratio R.

(a) Find the exact value of R.

[3]

The amount of money after one year

$$= P\left(1+\frac{6\%}{2}\right)^{(2)(1)}$$ $FV = PV\left(1+\frac{r\%}{k}\right)^{kn}$ (M1)

$$\therefore R = \left(1+\frac{6\%}{2}\right)^{(2)(1)}$$ $R = \frac{FV}{PV}$ (A1)

$R = 1.0609$ (A1)

It is given that there is no further deposit to or any withdrawal from the account.

(b) Find the year in which the amount of money in Judy's account will become double the amount she invested.

[3]

$FV = 2P$ Set up an equation

$$\therefore P\left(1+\frac{6\%}{2}\right)^{2n} = 2P$$ Correct equation (A1)

$$\left(1+\frac{6\%}{2}\right)^{2n} = 2$$

$$\left(1+\frac{6\%}{2}\right)^{2n} - 2 = 0$$

By considering the graph of

$y = \left(1+\frac{6\%}{2}\right)^{2n} - 2$, the horizontal intercept is

11.724886. GDC approach (M1)

$\therefore$ The required year is 2035. (A1)

It is given that $P = 22000$.

(c) (i) Find the amount that Judy will have in her account after 3 years.

[3]

The amount

$= PV\left(1+\frac{r\%}{k}\right)^{kn}$ — $FV = PV\left(1+\frac{r\%}{k}\right)^{kn}$ (M1)

$= 22000\left(1+\frac{6\%}{2}\right)^{(2)(3)}$ — $r = 6$, $k = 2$ & $n = 3$ (A1)

$= \$26269.15052$

$= \$26300$ (A1)

(ii) Hence, find the interest that Judy can earn after 3 years.

[2]

The interest

$= 26269.15052 - 22000$ — $I = FV - PV$ (M1)

$= \$4269.150524$

$= \$4270$ (A1)

Starting from 1st January 2024, Judy's friend Cady invests $\$16000$ in an account that pays a nominal annual interest rate of 8%, compounded **monthly**. It is given that the amount of money in Cady's account will exceed that in Judy's account after several years.

(d) Find the minimum number of complete years when the amount of money in Cady's account exceeds that in Judy's account.

[3]

Let t be the number of years required for the amount of money in Cady's account exceeding that in Judy's account.

$16000\left(1+\frac{8\%}{12}\right)^{12t} > 22000\left(1+\frac{6\%}{2}\right)^{2t}$ — Correct inequality (A1)

$16000\left(1+\frac{8\%}{12}\right)^{12t} - 22000\left(1+\frac{6\%}{2}\right)^{2t} > 0$

By considering the graph of

$y = 16000\left(1+\frac{8\%}{12}\right)^{12t} - 22000\left(1+\frac{6\%}{2}\right)^{2t}$, the

graph is above the horizontal axis when $t > 15.446241$. — GDC approach (M1)

$\therefore$ The minimum number of complete years is 16. (A1)

(e) Find the year when the sum of the amount of money in Cady's account and that in Judy's account first reaches $\$70000$.

[3]

Let T be the number of years required for the sum of the amount of money in Cady's account and that in Judy's account first reaches $\$70000$.

$$16000\left(1+\frac{8\%}{12}\right)^{12T}+22000\left(1+\frac{6\%}{2}\right)^{2T}\geq 70000$$ Correct inequality (A1)

$$16000\left(1+\frac{8\%}{12}\right)^{12T}+22000\left(1+\frac{6\%}{2}\right)^{2T}-70000\geq 0$$

By considering the graph of

$$y=16000\left(1+\frac{8\%}{12}\right)^{12T}+22000\left(1+\frac{6\%}{2}\right)^{2T}-70000$$

, the graph is above the horizontal axis when $T>8.9489625$. GDC approach (M1)

$\therefore$ The required year is 2032. (A1)

Exercise 1.6

On 1st January 2025, April invests $\$P$ in an account that pays a nominal annual interest rate of 8%, compounded **quarterly**. The amount of money in her account at the end of each year follows a geometric sequence with common ratio R.

(a) Find the exact value of R.

[3]

It is given that there is no further deposit to or any withdrawal from the account.

(b) Find the year in which the amount of money in April's account will become 2.5 times the amount she invested.

[3]

It is given that $P = 10000$.

(c) (i) Find the amount that April will have in her account after 4 years.

[3]

(ii) Hence, find the interest that April can earn after 4 years.

[2]

Starting from 1st January 2025, April's sister Bea invests $\$15000$ in an account that pays a nominal annual interest rate of 3%, compounded **half-yearly**. It is given that the amount of money in April's account will exceed that in Bea's account after several years.

(d) Find the minimum number of complete years when the amount of money in April's account exceeds that in Bea's account.

[3]

(e) Find the year when the sum of the amount of money in April's account and that in Bea's account first reaches $\$30500$.

[3]

4 Binomial Theorem

Important Notes

Properties of factorials and binomial coefficients:

1. $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$: n factorial
2. $0! = 1$
3. $C_r^n = \dfrac{n!}{r!(n-r)!}$: Binomial coefficient
4. $C_0^n = C_n^n = 1$
5. $C_1^n = C_{n-1}^n = n$
6. $C_r^n = C_{n-r}^n$ for $0 \le r \le n$, r, $n \in \mathbb{Z}^+$
7. $C_2^n = \dfrac{(n)(n-1)}{(2)(1)}$ can be expressed as a fraction in which both numerator and denominator are the product of two numbers, start descending from n and 2 respectively
8. $C_3^n = \dfrac{(n)(n-1)(n-2)}{(3)(2)(1)}$ can be expressed as a fraction in which both numerator and denominator are the product of three numbers, start descending from n and 3 respectively

Pascal triangle for finding binomial coefficients:

$n=0$	1
$n=1$	1 1
$n=2$	1 2 1
$n=3$	1 3 3 1
$n=4$	1 4 6 4 1
$n=5$	1 5 10 10 5 1
$\vdots$	$\vdots$
n	1 C_1^n C_2^n $\cdots$ C_{n-2}^n C_{n-1}^n 1

Properties of binomial theorem:

1. $(a+b)^n = a^n + C_1^n a^{n-1} b^1 + C_2^n a^{n-2} b^2 + \cdots + C_r^n a^{n-r} b^r + \cdots + C_{n-1}^n a^1 b^{n-1} + b^n$: Binomial theorem for $0 \le r \le n$, r, $n \in \mathbb{Z}^+$
2. $C_r^n a^{n-r} b^r$: General term (Term in b^r)

Properties of extended binomial theorem:

1. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$: Extended binomial theorem for $n \in \mathbb{Q}$, valid only when $-1 < x < 1$

2. $(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n = a^n\left[1 + n\left(\frac{b}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{b}{a}\right)^3 + \cdots\right]$ for $n \in \mathbb{Q}$, valid only when $-1 < \frac{b}{a} < 1$

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Example 1.7

The binomial expansion of $(1+px)^n$ is $1+24x+qx^2+\cdots+4096x^6$, $n\in\mathbb{Z}^+$, p, $q\in\mathbb{R}$. Find the values of n, p and q.

[7]

$(1+px)^n$

$=1^n+C_1^n1^{n-1}(px)^1+C_2^n1^{n-2}(px)^2+\cdots+(px)^n$ — Binomial theorem (M1)

$=1+(n)(1)(px)+\left(\frac{(n)(n-1)}{(2)(1)}\right)(1)(p^2x^2)+\cdots+p^nx^n$ — $C_1^n=n$ & $C_2^n=\frac{n(n-1)}{2}$ (A1)

$=1+npx+\frac{n(n-1)p^2}{2}x^2+\cdots+p^nx^n$

The term of the largest power of x is $4096x^6$.

$\therefore n=6$ — (A1)

$npx=24x$

$\therefore 6px=24x$ — Correct equation (A1)

$p=4$ — (A1)

$\frac{n(n-1)p^2}{2}x^2=qx^2$

$\therefore \frac{6(6-1)4^2}{2}x^2=qx^2$ — Correct equation (A1)

$q=240$ — (A1)

Exercise 1.7

The binomial expansion of $(1+px)^n$ is $1+\frac{10}{3}x+\frac{40}{9}x^2+qx^3+\cdots+p^nx^n$, $n\in\mathbb{Z}^+$, p, $q\in\mathbb{R}$. Find the values of n, p and q.

[7]

Example 1.8

The binomial expansion of $\sqrt{1+px}+\dfrac{1}{3+x}$, $p\in\mathbb{R}^{+}$, in ascending powers of x as far as the term in x^2, is $q+\dfrac{8}{9}x+rx^2$, q, $r\in\mathbb{R}$.

(a) Find the values of p, q and r.

[8]

$$\sqrt{1+px}+\frac{1}{3+x}$$

$$=(1+px)^{\frac{1}{2}}+(3+x)^{-1}$$ Negative & fractional indices (M1)

$$=(1+px)^{\frac{1}{2}}+3^{-1}\left(1+\frac{x}{3}\right)^{-1}$$

$$=\left[1+\left(\frac{1}{2}\right)(px)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(px)^2+\cdots\right]$$

Extended binomial theorem (M1)

$$+\frac{1}{3}\left[1+(-1)\left(\frac{x}{3}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{3}\right)^2+\cdots\right]$$

$$=\left[1+\frac{1}{2}px-\frac{1}{8}p^2x^2+\cdots\right]+\frac{1}{3}\left[1-\frac{1}{3}x+\frac{1}{9}x^2+\cdots\right]$$ $a+bx+cx^2$ (A1)

$$=\left[1+\frac{1}{2}px-\frac{1}{8}p^2x^2+\cdots\right]+\left[\frac{1}{3}-\frac{1}{9}x+\frac{1}{27}x^2+\cdots\right]$$

$$=\frac{4}{3}+\left(\frac{1}{2}p-\frac{1}{9}\right)x+\left(\frac{1}{27}-\frac{1}{8}p^2\right)x^2+\cdots$$

$$\therefore q=\frac{4}{3}$$ (A1)

$$\frac{1}{2}p-\frac{1}{9}=\frac{8}{9}$$ Compare coefficient of x (M1)

$$\frac{1}{2}p=1$$

$$p=2$$ (A1)

$$r=\frac{1}{27}-\frac{1}{8}(2)^2$$ Compare coefficient of x^2 (M1)

$$r=-\frac{25}{54}$$ (A1)

(b) Find the restriction on x such that this expansion is valid.

[2]

This expansion is valid when $-1 < px < 1$ and

$-1 < \frac{x}{3} < 1$. Compound inequality (M1)

$\therefore -1 < 2x < 1$ and $-1 < \frac{x}{3} < 1$

$-\frac{1}{2} < x < \frac{1}{2}$ and $-3 < x < 3$

$\therefore -\frac{1}{2} < x < \frac{1}{2}$ (A1)

Exercise 1.8

The binomial expansion of $\sqrt{4-x}-\frac{1}{1+px}$, $p\in\mathbb{R}^{+}$, in ascending powers of x as far as the term in x^2, is $q+rx-\frac{65}{64}x^2$, q, $r\in\mathbb{R}$.

(a) Find the values of p, q and r.

[8]

(b) Find the restriction on x such that this expansion is valid.

[2]

Example 1.9

Consider the expansion of $x^3(2+x^2)^n$, $n \in \mathbb{Z}^+$. The coefficient of x^5 is 1024.

(a) Find n.

[6]

The general term

$= x^3 \cdot C_r^n (2)^{n-r} (x^2)^r$ — $C_r^n a^{n-r} b^r$ (M1)

$= x^3 \cdot C_r^n 2^{n-r} x^{2r}$

$= C_r^n 2^{n-r} x^{3+2r}$ — Term in x^{3+2r} (A1)

Consider the term in x^5.

$\therefore 3+2r=5$ — Correct equation (A1)

$2r=2$

$r=1$ — $r=1$ (A1)

The coefficient of x^5 is 1024

$\therefore C_1^n 2^{n-1} = 1024$ — Correct equation (A1)

$n \cdot 2^{n-1} = 1024$

$n \cdot 2^{n-1} - 1024 = 0$

By considering the graph of $y = n \cdot 2^{n-1} - 1024$,

the horizontal intercept is 8.

$\therefore n=8$ — (A1)

(b) Hence, find the coefficient of x^7.

[2]

$x^3(2+x^2)^8$

$= x^3\left[\cdots + C_2^8 2^{8-2}(x^2)^2 + \cdots\right]$ — Binomial theorem (M1)

$= x^3\left[\cdots + 1792x^4 + \cdots\right]$

$= \cdots + 1792x^7 + \cdots$

Thus, the coefficient of x^7 is 1792. — (A1)

Exercise 1.9

Consider the expansion of $\dfrac{(1+x^3)^n}{2x^2}$, $n \in \mathbb{Z}^+$. The coefficient of x^4 is 3.

(a) Find n.

[6]

(b) Hence, find the coefficient of x^7.

[2]

5 Proofs and Identities

Important Notes

Identity of x: The equivalence of two expressions on two sides of the identity sign $\equiv$, for all values of x

Useful identities and expressions for proofs:

1. $(a+b)^2 \equiv a^2 + 2ab + b^2$
2. $(a-b)^2 \equiv a^2 - 2ab + b^2$
3. $(a+b)(a-b) \equiv a^2 - b^2$
4. x & $x+1$: Two consecutive integers, where $x \in \mathbb{Z}$
5. $2x+1$ & $2x+3$: Two consecutive odd numbers, where $x \in \mathbb{Z}$
6. $2x$ & $2x+2$: Two consecutive even numbers, where $x \in \mathbb{Z}$
7. aN: A multiple of a, where a, $N \in \mathbb{Z}$

Proof by counter example: Use an example to disprove a statement

Steps of proving that a statement is true by contradiction:

1. Assume that the opposite of the statement is true
2. Continue the proof until the new finding contradicts with some facts/assumptions
3. Conclude that the statement is true

Example 1.10

(a) Show that $(5n)^2+(5n+5)^2=50n^2+50n+25$, $n\in\mathbb{Z}$.

[2]

L.H.S.

$=(5n)^2+(5n+5)^2$ — Starts from L.H.S. (M1)

$=25n^2+25n^2+50n+25$ — $(a+b)^2\equiv a^2+2ab+b^2$ (A1)

$=50n^2+50n+25$

$=$R.H.S.

$\therefore (5n)^2+(5n+5)^2=50n^2+50n+25$ — (AG)

(b) Hence, or otherwise, prove that the sum of the squares of any two consecutive multiples of 5 is odd.

[3]

$5n$ and $5n+5$ are consecutive multiples of 5. — Consecutive multiples (R1)

$(5n)^2+(5n+5)^2=50n^2+50n+25$ — Proved in (a) (A1)

Also, $50n^2+50n+25$ is an odd integer. — $50n^2+50n$ is even (R1)

Thus, the sum of the squares of any two consecutive multiples of 5 is odd. — (AG)

Exercise 1.10

(a) Show that $(3n)^2 + (3n+3)^2 = 18n^2 + 18n + 9$, $n \in \mathbb{Z}$.

[2]

(b) Hence, or otherwise, prove that the sum of the squares of any two consecutive multiples of 3 is divisible by 9.

[3]

Example 1.11

(a) By using a counter example, show that $(m+n)^3 < m^3 + n^3$ is not always true, m, $n \in \mathbb{R}$.

[2]

Let $m = 1$ and $n = 2$. — Example of m & n (A1)

$(m+n)^3$

$= (1+2)^3$

$= 27$

$m^3 + n^3$

$= 1^3 + 2^3$ — Substitution (M1)

$= 9$

$\therefore (m+n)^3 > m^3 + n^3$

Thus, $(m+n)^3 < m^3 + n^3$ is not always true. — (AG)

(b) Prove by contradiction that $m^2 + 12n + 17 \neq 0$, m, $n \in \mathbb{Z}$.

[6]

Assume that $m^2 + 12n + 17 = 0$ for some m, $n \in \mathbb{Z}$. — Assumption (M1)

$m^2 = -12n - 17$

$m^2 = -12n - 18 + 1$

$m^2 = 2(-6n - 9) + 1$

Thus, m^2 is odd.

$\therefore m$ is also odd. — m^2 & m are odd (R1)

Let $m = 2k + 1$, $k \in \mathbb{Z}$. — $m = 2k + 1$ (M1)

$(2k+1)^2 = -12n - 17$

$4k^2 + 4k + 1 = -12n - 17$ — $(a+b)^2 = a^2 + 2ab + b^2$ (M1)

$4k^2 + 4k = -12n - 18$

$2k^2 + 2k = -6n - 9$

$2k^2 + 2k = -6n - 10 + 1$

$2(k^2 + k) = 2(-3n - 15) + 1$ — $2k_1 = 2k_2 + 1$ (A1)

As the right hand side $2(-3n-15)+1$ is odd, the left hand side $2(k^2+k)$ is also odd, which contradicts with the fact that $2(k^2+k)$ is even. — Contradiction (R1)

Thus, $m^2 + 12n + 17 \neq 0$, m, $n \in \mathbb{Z}$. — (AG)

Exercise 1.11

(a) By using a counter example, show that $|p+q|>|p|+|q|$ is not always true, where p, $q \in \mathbb{R}$.

[2]

(b) Prove by contradiction that the equation $4x^3-12x-17=0$ has no integer roots.

[4]

6 Mathematical Induction

Important Notes

Steps of proving a statement is true by mathematical induction:

1. Define $P(n)$ to be the statement to be proved
2. Prove that $P(n)$ is true when $n=1$
3. Assume that $P(n)$ is true when $n=k$
4. Prove that $P(n)$ is true when $n=k+1$
5. Conclude that $P(n)$ is true for all the values of n in the domain

Useful identities and expressions for proofs:

1. $(a+b)^2 \equiv a^2+2ab+b^2$
2. $(a-b)^2 \equiv a^2-2ab+b^2$
3. $(a+b)(a-b) \equiv a^2-b^2$
4. $\sum_{r=1}^{n} u_r = u_1+u_2+u_3+\cdots+u_{n-1}+u_n$: Summation sign
5. aN : A multiple of a, where a, $N \in \mathbb{Z}$

Example 1.12

Prove by mathematical induction that $\sum_{r=1}^{n} r(r+2) = \frac{n(n+1)(2n+7)}{6}$, $n \in \mathbb{Z}^+$.

[7]

Let $P(n)$: $\sum_{r=1}^{n} r(r+2) = \frac{n(n+1)(2n+7)}{6}$

When $n=1$,

L.H.S.

$= \sum_{r=1}^{1} r(r+2)$

$= 3$

R.H.S.

$= \frac{1(1+1)(2(1)+7)}{6}$

$= 3$

L.H.S. = R.H.S.

Thus, the statement is true when $n=1$. — Case when $n=1$ (R1)

Assume that the statement is true when $n=k$. — Assumption (M1)

$\sum_{r=1}^{k} r(r+2) = \frac{k(k+1)(2k+7)}{6}$

When $n=k+1$,

$\sum_{r=1}^{k+1} r(r+2)$

$= \sum_{r=1}^{k} r(r+2) + (k+1)(k+1+2)$ — Separate into two terms (M1)

$= \frac{k(k+1)(2k+7)}{6} + \frac{6(k+1)(k+3)}{6}$ — Apply the assumption (A1)

$= \frac{k+1}{6}[k(2k+7)+6(k+3)]$ — $\frac{k+1}{6}[\cdots]$ (A1)

$= \frac{k+1}{6}(2k^2+13k+18)$

$= \frac{(k+1)(k+2)(2k+9)}{6}$ — $(k+2)(2k+9)$ (A1)

$= \frac{(k+1)((k+1)+1)(2(k+1)+7)}{6}$

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n \in \mathbb{Z}^+$. — (R1)

Exercise 1.12

Prove by mathematical induction that $\frac{0}{1!}+\frac{1}{2!}+\frac{2}{3!}+\cdots+\frac{n-1}{n!}=1-\frac{1}{n!},\ n\in\mathbb{Z}^{+}$.

[6]

Example 1.13

Prove by mathematical induction that $18^n - 1$ is divisible by 17, $n \in \mathbb{Z}^+$.

[6]

Let $P(n)$: $18^n - 1$ is divisible by 17	
When $n = 1$,	
$18^1 - 1$	
$= 17$	
$= 17(1)$ which is divisible by 17	
Thus, the statement is true when $n = 1$.	Case when $n = 1$ (R1)
Assume that the statement is true when $n = k$.	Assumption (M1)
$18^k - 1 = 17M$, $M \in \mathbb{Z}$	
When $n = k + 1$,	
$18^{k+1} - 1$	
$= 18 \cdot 18^k - 1$	$18^{k+1} = 18^k \cdot 18^1$ (M1)
$= 18(17M + 1) - 1$	Apply the assumption (A1)
$= 306M + 18 - 1$	
$= 306M + 17$	
$= 17(18M + 1)$ which is divisible by 17	$17(18M + 1)$ (A1)
Thus, the statement is true when $n = k + 1$.	
Therefore, the statement is true for all $n \in \mathbb{Z}^+$.	(R1)

Exercise 1.13

Prove by mathematical induction that $3^{2n+1}+1$ is divisible by 4, $n\in\mathbb{Z}^+$.

[6]

7 Systems of Equations

Important Notes

Common systems of equations:

1. $\begin{cases} ax+by=c \\ dx+ey=f \end{cases}$: 2×2 system, $a, b, c, d, e, f \in \mathbb{R}$
2. $\begin{cases} ax+by+cz=d \\ ex+fy+gz=h \\ ix+jy+kz=l \end{cases}$: 3×3 system, $a, b, c, d, e, f, g, h, i, j, k, l \in \mathbb{R}$

Row operations of a system with row R_i:

1. Multiply the constant k to the row R_i (kR_i)
2. Add the row R_i to the row R_j (R_i+R_j)
3. Add the multiple of the row R_i to the row R_j (kR_i+R_j)

Number of solutions of a system with the last row $az=b$ after row operation:

1. The system has a unique solution if $a\neq0$
2. The system has no solution if $a=0$ and $b\neq0$
3. The system has infinite number of solutions if $a=0$ and $b=0$

Notes on GDC		
TEXAS TI-84 Plus CE	**TEXAS TI-Nspire CX**	**CASIO fx-CG50**
2nd x⁻¹ →EDIT to input the coefficient matrix →MATH→rref(to perform row operations for the coefficient matrix	templates→Matrix to input the coefficient matrix menu→Matrix & Vector →Reduced Row-Echelon Form to perform row operations for the coefficient matrix	F3 to input the coefficient matrix OPTN→F2→F6→F5 to perform row operations for the coefficient matrix

Example 1.14

Consider the following system of equations where a, $b \in \mathbb{R}$:

$$x - 7y + 2z = -3$$
$$3x + 12y - 5z = 13$$
$$-4x - 5y + az = b$$

(a) Find the conditions of a and b for which

(i) the system has a unique solution; [4]

$$\begin{cases} x - 7y + 2z = -3 \\ 3x + 12y - 5z = 13 \\ -4x - 5y + az = b \end{cases}$$

$$\rightarrow \begin{cases} x - 7y + 2z = -3 \\ 33y - 11z = 22 \\ -33y + (a+8)z = b - 12 \end{cases} \quad (R_2 - 3R_1 \; \& R_3 + 4R_1)$$ Row operations (M1)(A1)

$$\rightarrow \begin{cases} x - 7y + 2z = -3 \\ 33y - 11z = 22 \quad (R_3 + R_2) \\ (a-3)z = b + 10 \end{cases}$$ $(a-3)z = b + 10$ (A1)

Thus, the system has a unique solution when $a - 3 \neq 0$ and $b + 10 \in \mathbb{R}$.

$\therefore a \neq 3$ and $b \in \mathbb{R}$ (A1)

(ii) the system has no solutions; [1]

The system has no solutions when $a - 3 = 0$ and $b + 10 \neq 0$.

$\therefore a = 3$ and $b \neq -10$ (A1)

(iii) the system has infinite number of solutions. [1]

The system has infinite number of solutions when $a - 3 = 0$ and $b + 10 = 0$.

$\therefore a = 3$ and $b = -10$ (A1)

(b) Find the solution of the system of equations when $a=4$ and $b=0$.

[3]

$$\begin{cases} x-7y+2z=-3 \\ 33y-11z=22 \\ (4-3)z=0+10 \end{cases}$$

$z=10$ (A1)

$33y-11(10)=22$

$33y=132$

$y=4$ (A1)

$x-7(4)+2(10)=-3$

$x=5$ (A1)

Exercise 1.14

Consider the following system of equations where a, $b \in \mathbb{R}$:

$$x + 3y - 2z = 3$$
$$-2x - y + z = -1$$
$$-x + 2y + az = b$$

(a) Find the conditions of a and b for which

(i) the system has a unique solution;

[4]

(ii) the system has no solutions;

[1]

(iii) the system has infinite number of solutions.

[1]

(b) Find the solution of the system of equations when $a = 3$ and $b = 6$.

[3]

8 Permutations and Combinations

Important Notes

Properties of factorials, combination coefficients and permutation coefficients:

1. $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$: n factorial
2. $0! = 1$
3. $C_r^n = \dfrac{n!}{r!(n-r)!}$: Combination coefficient that represents the number of ways to select r objects from n different objects without regarding the order of the arrangement
4. $C_r^n = C_{n-r}^n$ for $0 \le r \le n$, r, $n \in \mathbb{Z}^+$
5. $P_r^n = \dfrac{n!}{(n-r)!}$: Permutation coefficient that represents the number of ways to select r objects from n different objects with regarding the order of the arrangement
6. $P_n^n = n!$ for $0 \le r = n$, r, $n \in \mathbb{Z}^+$

Example 1.15

Two boys and five girls are standing in a straight line. Find the number of different arrangements if two boys

(a) must stand with each other;

[3]

The number of different arrangements
$= 2! \times 6!$ — $2!$ (A1) & $6!$ (A1)
$= 1440$ — (A1)

(b) must not stand with each other;

[2]

The number of different arrangements
$= 7! - 1440$ — Complementary case (A1)
$= 3600$ — (A1)

(c) must stand with each other and both of them stand on either end.

[3]

The number of different arrangements
$= 2! \times 5! \times 2$ — $2! \times 5!$ (A1) & two cases (A1)
$= 480$ — (A1)

Exercise 1.15

In an interview, four male interviewees and six female interviewees are asked to sit on a row of ten seats outside the interview room. Find the number of ways if

(a) the interviewees of the same gender must sit together; [3]

(b) male interviewees must sit together; [3]

(c) male interviewees must not sit together. [2]

Example 1.16

In order to form a team to join a debate competition, a team of four is to be selected from a group of six boys and six girls from a debate club.

(a) Find the number of different possible teams that could be selected.

[2]

The number of different possible teams

$= C_4^{6+6}$ C_4^{12} (A1)

$= 495$ (A1)

(b) Find the number of different teams that could be selected if the team includes

(i) one boy and three girls;

[2]

The number of different possible teams

$= C_1^6 \times C_3^6$ C_1^6 or C_3^6 (A1)

$= 120$ (A1)

(ii) at most two boys.

[2]

The number of different possible teams

$= C_0^6 \times C_4^6 + C_1^6 \times C_3^6 + C_2^6 \times C_2^6$ Three cases (A1)

$= 360$ (A1)

Exercise 1.16

From a group of eight male students and two female students, five students are selected to form a group.

(a) Determine how many possible groups can be formed. [2]

(b) Determine how many groups can be formed consisting of

(i) three male students and two female students; [2]

(ii) at least one male student. [2]

9 Complex Numbers

Important Notes

Terminologies of complex numbers:

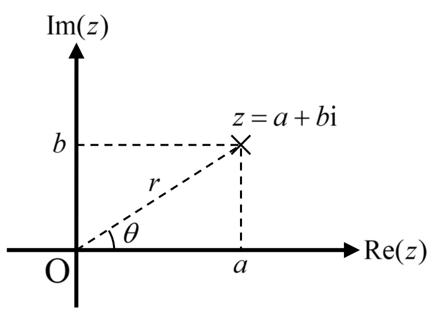
1. $\text{i}=\sqrt{-1}$: Imaginary unit
2. $z=a+b\text{i}$: Complex number in Cartesian form
3. $a=\text{Re}(z)$: Real part of $z=a+b\text{i}$
4. $b=\text{Im}(z)$: Imaginary part of $z=a+b\text{i}$
5. $z^*=a-b\text{i}$: Conjugate of $z=a+b\text{i}$
6. $|z|=\sqrt{a^2+b^2}$: Modulus of $z=a+b\text{i}$
7. $\arg(z)=\arctan\frac{b}{a}$: Argument of $z=a+b\text{i}$

Properties of i for $N\in\mathbb{Z}$:

1. $\text{i}=\text{i}^5=\text{i}^9=\cdots=\text{i}^{4N+1}=\text{i}$
2. $\text{i}^2=\text{i}^6=\text{i}^{10}=\cdots=\text{i}^{4N+2}=-1$
3. $\text{i}^3=\text{i}^7=\text{i}^{11}=\cdots=\text{i}^{4N+3}=-\text{i}$
4. $\text{i}^4=\text{i}^8=\text{i}^{12}=\cdots=\text{i}^{4N}=1$

Properties of Argand diagram:

1. Real axis: Horizontal axis
2. Imaginary axis: Vertical axis
3. $r=|z|=\sqrt{a^2+b^2}$: Modulus of $z=a+b\text{i}$
4. $\theta=\arg(z)=\arctan\frac{b}{a}$: Argument of $z=a+b\text{i}$



Forms of complex numbers:

1. $z=a+b\text{i}$: Cartesian form
2. $z=r(\cos\theta+\text{i}\sin\theta)=r\text{cis}\theta$: Modulus-argument (polar) form
3. $z=re^{\text{i}\theta}$: Euler (exponential) form

Properties of moduli and arguments of complex numbers z_1 and z_2:

1. $|z_1 z_2| = |z_1||z_2|$
2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
3. $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
4. $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
5. $\arg(z_1^n) = n \arg z_1$

Applications to polynomials and useful expressions:

1. If $z = a + b\mathrm{i}$ is a root of the polynomial equation $p(z) = 0$, then $z^* = a - b\mathrm{i}$ is also a root of $p(z) = 0$
2. $zz^* = (a + b\mathrm{i})(a - b\mathrm{i}) = a^2 + b^2$

Roots of complex numbers and De Moivre's theorem:

1. $z = r^{\frac{1}{n}} \mathrm{cis}\left(\frac{\theta + 2k\pi}{n}\right)$, $k = 0, 1, 2, \cdots, n-1$: n distinct complex roots of the equation $z^n = r\mathrm{cis}\theta$
2. $(r\mathrm{cis}\theta)^n = r^n \mathrm{cis} n\theta$ & $(r\mathrm{e}^{\theta\mathrm{i}})^n = r^n \mathrm{e}^{n\theta\mathrm{i}}$: De Moivre's theorem

Notes on GDC		
TEXAS TI-84 Plus CE	**TEXAS TI-Nspire CX**	**CASIO fx-CG50**
mode→re^(θi) on the 8th row to express a complex number in its Euler form to find its modulus and argument →a+bi on the 8th row to express a complex number in its Cartesian form to find its real part and imaginary part	menu→Number →Complex Number Tools →Convert to Polar to express a complex number in its Euler form to find its modulus and argument →Convert to Rectangular to express a complex number in its Cartesian form to find its real part and imaginary part	SHIFT→MENU→F3 on the row Complex Mode to express a complex number in its Euler form to find its modulus and argument →F2 on the row Complex Mode to express a complex number in its Cartesian form to find its real part and imaginary part

Example 1.17

Three of the roots of the equation $z^4 - 13z^3 + 59z^2 - 117z + k = 0$, $z \in \mathbb{C}$, $k \in \mathbb{R}$ are α, 2α and $2 - \mathrm{i}$, $\alpha \in \mathbb{R}$.

(a) Using the sum of all the roots of the equation to find α.

[4]

$2 - \mathrm{i}$ is one of the roots

$\therefore 2 + \mathrm{i}$ is also one of the roots — Conjugate (A1)

Sum of roots $= -\dfrac{-13}{1}$ — $-\dfrac{a_3}{a_4}$ (M1)

$\therefore \alpha + 2\alpha + (2 - \mathrm{i}) + (2 + \mathrm{i}) = -\dfrac{-13}{1}$ — Correct equation (A1)

$3\alpha + 4 = 13$

$3\alpha = 9$

$\alpha = 3$ — (A1)

(b) Hence, find k.

[3]

Product of roots $= (-1)^4 \dfrac{k}{1}$ — $(-1)^4 \dfrac{a_0}{a_4}$ (M1)

$\therefore (3)(6)(2 - \mathrm{i})(2 + \mathrm{i}) = (-1)^4 \dfrac{k}{1}$ — Correct equation (A1)

$18(2^2 - \mathrm{i}^2) = k$

$18(4 - (-1)) = k$

$k = 90$ — (A1)

Exercise 1.17

Two of the roots of the equation $z^4 + kz^3 + 8z^2 + 80z + 400 = 0$, $z \in \mathbb{C}$, $k \in \mathbb{R}$ are $\alpha + 2\text{i}$ and $2 - 4\text{i}$, $\alpha \in \mathbb{R}$, $\alpha < 0$.

(a) Using the product of all the roots of the equation to find α.

[4]

(b) Hence, find k.

[3]

Example 1.18

(a) (i) Use the binomial theorem to expand $(\cos\theta + \mathrm{i}\sin\theta)^3$.

[2]

$(\cos\theta + \mathrm{i}\sin\theta)^3$

$= \cos^3\theta + C_1^3\cos^2\theta(\mathrm{i}\sin\theta)^1$
$+ C_2^3\cos\theta(\mathrm{i}\sin\theta)^2 + (\mathrm{i}\sin\theta)^3$ — Binomial theorem (A1)

$= \cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta$
$+ 3\cos\theta(\mathrm{i}^2\sin^2\theta) + \mathrm{i}^3\sin^3\theta$

$= \cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta$
$-3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta$ — (A1)

(ii) Hence, use De Moivre's theorem to show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

[3]

$(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos^3\theta$
$+ 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta$

$\cos 3\theta + \mathrm{i}\sin 3\theta = \cos^3\theta$
$+ 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta$ — De Moivre's theorem (M1)

$\therefore \mathrm{i}\sin 3\theta = 3\mathrm{i}\cos^2\theta\sin\theta - \mathrm{i}\sin^3\theta$ — Compare imaginary parts (M1)

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

$\sin 3\theta = 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta$ — $\cos^2\theta = 1 - \sin^2\theta$ (A1)

$\sin 3\theta = 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ — (AG)

It is given that $\cos 3\theta = \alpha\cos^3\theta - 3\cos\theta$, $\alpha \in \mathbb{R}$.

(iii) Find α.

[3]

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ — Compare real parts (M1)

$\cos 3\theta = \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$ — $\sin^2\theta = 1 - \cos^2\theta$ (A1)

$\cos 3\theta = \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\therefore \alpha = 4$ — (A1)

Let $z = r(\cos\beta + \mathrm{i}\sin\beta)$, $\beta \in \mathbb{R}$, be the solution of $z^3 = \cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)$ which has the smallest positive argument.

(b) Find the values of r and β.

[4]

$$z^3 = \cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)$$

$$z = \cos\left(\frac{\frac{\pi}{4} + 2\pi k}{3}\right) + \mathrm{i}\sin\left(\frac{\frac{\pi}{4} + 2\pi k}{3}\right) \quad (k = 0, 1, 2)$$ 3 complex roots (M1)

$$z = \cos\left(\frac{\pi}{12} + \frac{2\pi k}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{12} + \frac{2\pi k}{3}\right)$$ $\frac{\pi}{12} + \frac{2\pi k}{3}$ (A1)

$\therefore r = 1$ (A1)

$$\beta = \frac{\pi}{12} + \frac{2\pi(0)}{3}$$

$\beta = \frac{\pi}{12}$ (A1)

(c) Hence, show that $8\sin^3\left(\frac{\pi}{12}\right) - 6\sin\left(\frac{\pi}{12}\right) + \sqrt{2} = 0$.

[3]

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\therefore \sin 3\left(\frac{\pi}{12}\right) = 3\sin\left(\frac{\pi}{12}\right) - 4\sin^3\left(\frac{\pi}{12}\right)$$ $\theta = \frac{\pi}{12}$ (M1)

$$\sin\left(\frac{\pi}{4}\right) = 3\sin\left(\frac{\pi}{12}\right) - 4\sin^3\left(\frac{\pi}{12}\right)$$ $3\left(\frac{\pi}{12}\right) = \frac{\pi}{4}$ (A1)

$$\frac{\sqrt{2}}{2} = 3\sin\left(\frac{\pi}{12}\right) - 4\sin^3\left(\frac{\pi}{12}\right)$$ $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ (A1)

$$4\sin^3\left(\frac{\pi}{12}\right) - 3\sin\left(\frac{\pi}{12}\right) + \frac{\sqrt{2}}{2} = 0$$

$$8\sin^3\left(\frac{\pi}{12}\right) - 6\sin\left(\frac{\pi}{12}\right) + \sqrt{2} = 0$$ (AG)

Let $w = R(\cos\lambda + \mathrm{i}\sin\lambda)$, $\lambda \in \mathbb{R}$, be the solution of $(w-2)^3 = \cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)$ which has the smallest positive argument.

(d) (i) Describe the geometric relationship between $z = r(\cos\beta + \mathrm{i}\sin\beta)$ and $w = R(\cos\lambda + \mathrm{i}\sin\lambda)$.

[1]

$w = R(\cos\lambda + \mathrm{i}\sin\lambda)$ is 2 units on the right of $z = r(\cos\beta + \mathrm{i}\sin\beta)$. (A1)

(ii) Show that $\mu = \frac{2e^{\frac{7\pi}{12}\mathrm{i}} + 1}{e^{\frac{7\pi}{12}\mathrm{i}}}$ is a root of $(w-2)^3 = \cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)$.

[3]

$(\mu - 2)^3$

$= \left(\frac{2e^{\frac{7\pi}{12}\mathrm{i}} + 1}{e^{\frac{7\pi}{12}\mathrm{i}}} - 2\right)^3$ Substitution (M1)

$= \left(2 + \frac{1}{e^{\frac{7\pi}{12}\mathrm{i}}} - 2\right)^3$

$= \left(\frac{1}{e^{\frac{7\pi}{12}\mathrm{i}}}\right)^3$

$= \frac{1}{e^{\frac{7\pi}{4}\mathrm{i}}}$ $\left(e^{\frac{7\pi}{12}\mathrm{i}}\right)^3 = e^{\frac{7\pi}{4}\mathrm{i}}$ (A1)

$= e^{-\frac{7\pi}{4}\mathrm{i}}$

$= \cos\left(-\frac{7\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{7\pi}{4}\right)$

$= \cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)$ $-\frac{7\pi}{4} + 2\pi = \frac{\pi}{4}$ (A1)

Thus, $\mu = \frac{2e^{\frac{7\pi}{12}\mathrm{i}} + 1}{e^{\frac{7\pi}{12}\mathrm{i}}}$ is a root of $(w-2)^3 = \cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)$. (AG)

Exercise 1.18

(a) (i) Use the binomial theorem to expand $(\cos\theta + \mathrm{i}\sin\theta)^4$.

[2]

(ii) Hence, use De Moivre's theorem to show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

[3]

It is given that $\sin 4\theta = 4\sin\theta\cos\theta + \alpha\sin^3\theta\cos\theta$, $\alpha \in \mathbb{R}$.

(iii) Find α.

[3]

Let $z = r(\cos\beta + \mathrm{i}\sin\beta)$, $\beta \in \mathbb{R}$, be the solution of $z^4 = \mathrm{i}$ which has the smallest positive argument.

(b) Find the values of r and β.

[4]

(c) Hence, show that $4\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)\left(1 - 2\sin^2\left(\frac{\pi}{8}\right)\right) - 1 = 0$.

[3]

Let $w = R(\cos\lambda + \mathrm{i}\sin\lambda)$, $\lambda \in \mathbb{R}$, be the solution of $(w+2\mathrm{i})^4 = \mathrm{i}$ which has the smallest positive argument.

(d) (i) Describe the geometric relationship between $z = r(\cos\beta + \mathrm{i}\sin\beta)$ and $w = R(\cos\lambda + \mathrm{i}\sin\lambda)$.

[1]

(ii) Show that $\mu = \dfrac{1 - 2\mathrm{i}e^{\frac{11\pi}{8}\mathrm{i}}}{e^{\frac{11\pi}{8}\mathrm{i}}}$ is a root of $(w+2\mathrm{i})^4 = \mathrm{i}$.

[3]

Example 1.19

Two complex numbers are defined as $w=27\left(\cos\frac{\pi}{15}-\mathrm{i}\sin\frac{\pi}{15}\right)$ and $z=3\left(\cos\frac{k\pi}{5}+\mathrm{i}\sin\frac{k\pi}{5}\right)$, $k\in\mathbb{Z}^+$.

(a) Find the modulus of wz.

[2]

The modulus of wz

$=|w||z|$ — $|wz|=|w||z|$ (M1)

$=(27)(3)$

$=81$ — (A1)

(b) Find the argument of wz, giving the answer in terms of k.

[2]

w

$=27\left(\cos\frac{\pi}{15}-\mathrm{i}\sin\frac{\pi}{15}\right)$

$=27\left(\cos\left(-\frac{\pi}{15}\right)+\mathrm{i}\sin\left(-\frac{\pi}{15}\right)\right)$

$\therefore \arg(w)=-\frac{\pi}{15}$

The argument of wz

$=\arg(w)+\arg(z)$ — $\arg(wz)=\arg(w)+\arg(z)$ (M1)

$=-\frac{\pi}{15}+\frac{k\pi}{5}$

$=-\frac{\pi}{15}+\frac{3k\pi}{15}$

$=\frac{(3k-1)\pi}{15}$ — (A1)

(c) Is it possible for wz to be on the real axis? Explain your answer.

[2]

$\arg(wz)$

$= \frac{(3k-1)\pi}{15}$

As $3k-1$ is not divisible by 3 nor 15, $\frac{(3k-1)\pi}{15}$

can never be a multiple of π. Indivisibility (R1)

$\therefore \arg(wz) \neq N\pi$, $N \in \mathbb{Z}$

Thus, wz can never be on the real axis. (A1)

Exercise 1.19

Two complex numbers are defined as $w=\frac{1}{8}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)$ and $z=\frac{1}{2}\left(\cos\frac{k\pi}{4}-\mathrm{i}\sin\frac{k\pi}{4}\right)$, $k\in\mathbb{Z}^{+}$.

(a) Find the modulus of wz.

[2]

(b) Find the argument of wz, giving the answer in terms of k.

[2]

(c) It is given that wz is on the imaginary axis. Find the minimum value of k.

[3]