Exercise 1.1

(a) The required hypotenuse

$= \sqrt{1107^2 + 4920^2}$ — Pythagoras' theorem (A1)

$= 5043 \text{ mm}$

$= 5.043 \times 10^3 \text{ mm}$ — $a = 5.043$ & $k = 3$ (A1)

(b) The required perimeter

$= 1107 + 4920 + 5043$ — The sum of 3 sides (A1)

$= 11070 \text{ mm}$

$= 11000 \text{ mm}$ — Round off to 2 sig. fig.

$= 1.1 \times 10^4 \text{ mm}$ — $a = 1.1$ & $k = 4$ (A1)

(c) The required area

$= \dfrac{(1107)(4920)}{2}$ — $\dfrac{\text{Base length} \times \text{Height}}{2}$ (A1)

$= 2723220 \text{ mm}^2$

$= 2723000 \text{ mm}^2$ — Round off to 4 sig. fig.

$= 2.723 \times 10^6 \text{ mm}^2$ — $a = 2.723$ & $k = 6$ (A1)

Exercise 1.2

(a)

$d=\frac{38}{10}-\frac{39}{10}$ — $d=u_2-u_1$

$d=-\frac{1}{10}$ — (A1)

(b)

$u_{12}=u_1+(12-1)d$ — $u_n=u_1+(n-1)d$

$u_{12}=\frac{39}{10}+(12-1)\left(-\frac{1}{10}\right)$ — Correct approach (A1)

$u_{12}=\frac{39}{10}-\frac{11}{10}$

$u_{12}=\frac{28}{10}$

$u_{12}=\frac{14}{5}$ — (A1)

(c)

$u_n=\frac{7}{10}$ — Set up an equation

$\therefore\frac{39}{10}+(n-1)\left(-\frac{1}{10}\right)=\frac{7}{10}$ — Correct equation (A1)

$-\frac{1}{10}(n-1)=-\frac{32}{10}$

$n-1=32$

$n=33$ — (A1)

(d) The sum of the first n terms

$=S_n$

$=\frac{n}{2}\left[2u_1+(n-1)d\right]$ — $S_n=\frac{n}{2}\left[2u_1+(n-1)d\right]$ (M1)

$=\frac{n}{2}\left[2\left(\frac{39}{10}\right)+(n-1)\left(-\frac{1}{10}\right)\right]$ — $u_1=\frac{39}{10}$ & $d=-\frac{1}{10}$ (A1)

$=\frac{n}{2}\left(\frac{78}{10}-\frac{1}{10}n+\frac{1}{10}\right)$

$=\frac{n}{2}\left(-\frac{1}{10}n+\frac{79}{10}\right)$

$=-\frac{1}{20}n^2+\frac{79}{20}n$ — (A1)

(e) The required sum

$= S_{10}$

$= -\frac{1}{20}(10)^2 + \frac{79}{20}(10)$ $n = 10$ (M1)

$= -5 + \frac{79}{2}$

$= \frac{69}{2}$ (A1)

Exercise 1.3

(a) $u_m = 0$ — Set up an equation

$\therefore 120 + (m-1)(-1.25) = 0$ — Correct equation (A1)

$-1.25(m-1) = -120$

$m - 1 = 96$

$m = 97$ — (A1)

(b) The sum of the first n terms

$= S_n$

$= \frac{n}{2}\left[2u_1 + (n-1)d\right]$ — $S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$ (M1)

$= \frac{n}{2}\left[2(120) + (n-1)(-1.25)\right]$

$= \frac{n}{2}(240 - 1.25n + 1.25)$

$= \frac{n(241.25 - 1.25n)}{2}$

By considering the graph of $y = \frac{n(241.25 - 1.25n)}{2}$, the coordinates of the maximum point are $(96.5, 5820.1563)$, and the graph passes through $(96, 5820)$ and $(97, 5820)$. — GDC approach (M1)

Thus, the maximum value is 5820. — (A1)

Exercise 1.4

(a) (i) $k\log x - \log x = \frac{1}{5}\log x - k\log x$ — $d = u_2 - u_1 = u_3 - u_2$ (A1)

$k - 1 = \frac{1}{5} - k$

$2k = \frac{6}{5}$

$k = \frac{3}{5}$ — (A1)

(ii) The common difference

$= \frac{3}{5}\log x - \log x$ — $d = u_2 - u_1$ (A1)

$= -\frac{2}{5}\log x$ — (A1)

(b) (i) $k\log x \div \log x = \frac{1}{5}\log x \div k\log x$ — $r = u_2 \div u_1 = u_3 \div u_2$ (A1)

$k = \frac{1}{5k}$

$5k^2 = 1$

$k^2 = \frac{1}{5}$

$k = \frac{1}{\sqrt{5}}$ or $k = -\frac{1}{\sqrt{5}}$ — (A1)(A1)

(ii) The sum of the first n terms

$= S_n$

$= \frac{u_1(1-r^n)}{1-r}$ — $S_n = \frac{u_1(1-r^n)}{1-r}$ (M1)

$= \frac{(\log x)\left(1-\left(-\frac{1}{\sqrt{5}}\right)^n\right)}{1-\left(-\frac{1}{\sqrt{5}}\right)}$ — $u_1 = \log x$ & $r = -\frac{1}{\sqrt{5}}$ (A1)

$= \frac{(\log x)\left(1-\left(-\frac{1}{5^{0.5}}\right)^n\right)}{1+\frac{1}{5^{0.5}}}$ — $\sqrt{5} = 5^{0.5}$ (M1)

$= \frac{(\log x)(1-(-5^{-0.5})^n)}{1+5^{-0.5}}$

$= \frac{1-(-1)^n 5^{-0.5n}}{1+5^{-0.5}} \log x$ — (A1)

(iii) $S_\infty = \frac{5+\sqrt{5}}{2}$ — Set up an equation

$\therefore \frac{\log x}{1-\frac{1}{\sqrt{5}}} = \frac{5+\sqrt{5}}{2}$ — Correct equation (A1)

$\log x = \frac{1}{2}(5+\sqrt{5})\left(1-\frac{1}{\sqrt{5}}\right)$

$\log x = \frac{1}{2}(5-\sqrt{5}+\sqrt{5}-1)$

$\log x = 2$ — Simplify the R.H.S. (A1)

$x = 10^2$

$x = 100$ — (A1)

Exercise 1.5

(a) $r=\frac{7}{16}\div\frac{7}{12}$ — $r=u_2\div u_1$

$r=\frac{3}{4}$ — (A1)

(b) $u_7=u_1\times r^{7-1}$ — $u_n=u_1\times r^{n-1}$

$u_7=\frac{7}{12}\times\left(\frac{3}{4}\right)^{7-1}$ — $u_1=\frac{7}{12}$ & $r=\frac{3}{4}$ (A1)

$u_7=0.1038208008$

$u_7=0.104$ — (A1)

(c) $u_n=\frac{189}{1024}$ — Set up an equation

$\therefore\frac{7}{12}\times\left(\frac{3}{4}\right)^{n-1}=\frac{189}{1024}$ — Correct equation (A1)

$\frac{7}{12}\times\left(\frac{3}{4}\right)^{n-1}-\frac{189}{1024}=0$

By considering the graph of

$y=\frac{7}{12}\times\left(\frac{3}{4}\right)^{n-1}-\frac{189}{1024}$, the horizontal intercept is 5.

$\therefore n=5$ — (A1)

(d) The sum of the first n terms

$=S_n$

$=\frac{u_1(1-r^n)}{1-r}$ — $S_n=\frac{u_1(1-r^n)}{1-r}$ (M1)

$=\frac{\frac{7}{12}\left(1-\left(\frac{3}{4}\right)^n\right)}{1-\frac{3}{4}}$ — $u_1=\frac{7}{12}$ & $r=\frac{3}{4}$ (A1)

$=\frac{7}{3}\left(1-\left(\frac{3}{4}\right)^n\right)$ — (A1)

(e) The required sum

$= S_{15}$

$= \frac{7}{3}\left(1-\left(\frac{3}{4}\right)^{15}\right)$ — $n = 15$ (M1)

$= 2.302151924$

$= 2.30$ — (A1)

(f) The common ratio is $\frac{3}{4}$ which is between -1 and 1. — (R1)

(g) $S_n < 2.3315$ — Set up an inequality

$\therefore \frac{7}{3}\left(1-\left(\frac{3}{4}\right)^{n}\right) < 2.3315$ — Correct inequality (A1)

$\frac{7}{3}\left(1-\left(\frac{3}{4}\right)^{n}\right) - 2.3315 < 0$

By considering the graph of $y = \frac{7}{3}\left(1-\left(\frac{3}{4}\right)^{n}\right) - 2.3315$, the graph is below the horizontal axis when $n < 24.850062$. — GDC approach (M1)

$\therefore$ The greatest value of n is 24. — (A1)

Exercise 1.6

(a) The amount of money after one year

$= P\left(1+\frac{8\%}{4}\right)^{(4)(1)}$ $\quad FV = PV\left(1+\frac{r\%}{k}\right)^{kn}$ (M1)

$\therefore R = \left(1+\frac{8\%}{4}\right)^{(4)(1)}$ $\quad R = \frac{FV}{PV}$ (A1)

$R = 1.08243216$ (A1)

(b) $FV = 2.5P$ Set up an equation

$\therefore P\left(1+\frac{8\%}{4}\right)^{4n} = 2.5P$ Correct equation (A1)

$\left(1+\frac{8\%}{4}\right)^{4n} = 2.5$

$\left(1+\frac{8\%}{4}\right)^{4n} - 2.5 = 0$

By considering the graph of

$y = \left(1+\frac{8\%}{4}\right)^{4n} - 2.5$, the horizontal intercept is

11.567792. GDC approach (M1)

$\therefore$ The required year is 2036. (A1)

(c) (i) The amount

$= PV\left(1+\frac{r\%}{k}\right)^{kn}$ $\quad FV = PV\left(1+\frac{r\%}{k}\right)^{kn}$ (M1)

$= 10000\left(1+\frac{8\%}{4}\right)^{(4)(4)}$ $\quad r = 8,\ k = 4\ \&\ n = 4$ (A1)

$= \$13727.85705$

$= \$13700$ (A1)

(ii) The interest

$= 13727.85705 - 10000$ $\quad I = FV - PV$ (M1)

$= \$3727.857051$

$= \$3730$ (A1)

(d) Let t be the number of years required for the amount of money in April's account exceeding that in Bea's account.

$$10000\left(1+\frac{8\%}{4}\right)^{4t} > 15000\left(1+\frac{3\%}{2}\right)^{2t}$$ Correct inequality (A1)

$$10000\left(1+\frac{8\%}{4}\right)^{4t} - 15000\left(1+\frac{3\%}{2}\right)^{2t} > 0$$

By considering the graph of

$y = 10000\left(1+\frac{8\%}{4}\right)^{4t} - 15000\left(1+\frac{3\%}{2}\right)^{2t}$, the graph is above the horizontal axis when $t > 8.2022693$. GDC approach (M1)

$\therefore$ The minimum number of complete years is 9. (A1)

(e) Let T be the number of years required for the sum of the amount of money in April's account and that in Bea's account first reaches $\$30500$.

$$10000\left(1+\frac{8\%}{4}\right)^{4T} + 15000\left(1+\frac{3\%}{2}\right)^{2T} \ge 30500$$ Correct inequality (A1)

$$10000\left(1+\frac{8\%}{4}\right)^{4T} + 15000\left(1+\frac{3\%}{2}\right)^{2T} - 30500 \ge 0$$

By considering the graph of

$y = 10000\left(1+\frac{8\%}{4}\right)^{4T} + 15000\left(1+\frac{3\%}{2}\right)^{2T} - 30500$, the graph is above the horizontal axis when $T > 3.9210584$. GDC approach (M1)

$\therefore$ The required year is 2028. (A1)

Exercise 1.7

$$(1+px)^n$$
$$=1^n+C_1^n1^{n-1}(px)^1+C_2^n1^{n-2}(px)^2+\cdots+(px)^n$$ Binomial theorem (M1)

$$=1+(n)(1)(px)+\left(\frac{(n)(n-1)}{(2)(1)}\right)(1)(p^2x^2)+\cdots+p^nx^n$$ $C_1^n=n$ & $C_2^n=\frac{n(n-1)}{2}$ (A1)

$$=1+npx+\frac{n(n-1)p^2}{2}x^2+\cdots+p^nx^n$$
$$npx=\frac{10}{3}x$$
$$p=\frac{10}{3n}$$ Make p the subject (A1)

$$\frac{n(n-1)p^2}{2}x^2=\frac{40}{9}x^2$$
$$\therefore\frac{n(n-1)}{2}\left(\frac{10}{3n}\right)^2=\frac{40}{9}$$ Substitution (M1)

$$\frac{n(n-1)}{2}\left(\frac{100}{9n^2}\right)=\frac{40}{9}$$
$$\frac{50(n-1)}{9n}=\frac{40}{9}$$
$$50n-50=40n$$
$$-50=-10n$$
$$n=5$$ (A1)

$$\therefore p=\frac{10}{3(5)}$$
$$p=\frac{2}{3}$$ (A1)

$$C_3^n1^{n-3}(px)^3=qx^3$$
$$\therefore C_3^5(1)\left(\frac{2}{3}\right)^3x^3=qx^3$$
$$q=\frac{80}{27}$$ (A1)

Exercise 1.8

(a) $\sqrt{4-x}-\dfrac{1}{1+px}$

$=(4-x)^{\frac{1}{2}}-(1+px)^{-1}$ — Negative & fractional indices (M1)

$=4^{\frac{1}{2}}\left(1-\dfrac{x}{4}\right)^{\frac{1}{2}}-(1+px)^{-1}$

$=2\left[1+\left(\dfrac{1}{2}\right)\left(-\dfrac{x}{4}\right)+\dfrac{\left(\dfrac{1}{2}\right)\left(-\dfrac{1}{2}\right)}{2!}\left(-\dfrac{x}{4}\right)^2+\cdots\right]$ — Extended binomial theorem (M1)

$-\left[1+(-1)(px)+\dfrac{(-1)(-2)}{2!}(px)^2+\cdots\right]$

$=2\left[1-\dfrac{1}{8}x-\dfrac{1}{128}x^2+\cdots\right]-\left[1-px+p^2x^2+\cdots\right]$ — $a+bx+cx^2$ (A1)

$=\left[2-\dfrac{1}{4}x-\dfrac{1}{64}x^2+\cdots\right]-\left[1-px+p^2x^2+\cdots\right]$

$=1+\left(p-\dfrac{1}{4}\right)x+\left(-p^2-\dfrac{1}{64}\right)x^2+\cdots$

$\therefore q=1$ — (A1)

$-p^2-\dfrac{1}{64}=-\dfrac{65}{64}$ — Compare coefficient of x^2 (M1)

$-p^2=-1$

$p^2=1$

$p=1$ — (A1)

$r=1-\dfrac{1}{4}$ — Compare coefficient of x (M1)

$r=\dfrac{3}{4}$ — (A1)

(b) This expansion is valid when $-1<-\dfrac{x}{4}<1$ and

$-1<px<1$. — Compound inequality (M1)

$\therefore -1<-\dfrac{x}{4}<1$ and $-1<x<1$

$4>x>-4$ and $-1<x<1$

$\therefore -1<x<1$ — (A1)

Exercise 1.9

(a) The general term

$= \frac{1}{2x^2} \cdot C_r^n (1)^{n-r} (x^3)^r$ — $C_r^n a^{n-r} b^r$ (M1)

$= \frac{1}{2x^2} \cdot C_r^n x^{3r}$

$= \frac{1}{2} C_r^n x^{3r-2}$ — Term in x^{3r-2} (A1)

Consider the term in x^4.

$\therefore 3r - 2 = 4$ — Correct equation (A1)

$3r = 6$

$r = 2$ — $r = 2$ (A1)

The coefficient of x^4 is 3

$\therefore \frac{1}{2} C_2^n = 3$ — Correct equation (A1)

$\frac{1}{2}\left(\frac{(n)(n-1)}{(2)(1)}\right) = 3$

$\frac{n(n-1)}{4} - 3 = 0$

By considering the graph of $y = \frac{n(n-1)}{4} - 3$, the horizontal intercept is 4.

$\therefore n = 4$ — (A1)

(b) $\frac{(1+x^3)^n}{2x^2}$

$= \frac{1}{2x^2}\left[\cdots + C_3^4 1^{4-3} (x^3)^3 + \cdots\right]$ — Binomial theorem (M1)

$= \frac{1}{2x^2}\left[\cdots + 4x^9 + \cdots\right]$

$= \cdots + 2x^7 + \cdots$

Thus, the coefficient of x^7 is 2. — (A1)

Exercise 1.10

(a) L.H.S.

$=(3n)^2+(3n+3)^2$ Starts from L.H.S. (M1)

$=9n^2+9n^2+18n+9$ $(a+b)^2 \equiv a^2+2ab+b^2$ (A1)

$=18n^2+18n+9$

$=$R.H.S.

$\therefore (3n)^2+(3n+3)^2=18n^2+18n+9$ (AG)

(b) $3n$ and $3n+3$ are consecutive multiples of 3. Consecutive multiples (R1)

$(3n)^2+(3n+3)^2=18n^2+18n+9$ Proved in (a) (A1)

Also, $18n^2+18n+9=9(2n^2+2n+1)$ which is divisible by 9. $9(2n^2+2n+1)$ (R1)

Thus, the sum of the squares of any two consecutive multiples of 3 is divisible by 9. (AG)

Exercise 1.11

(a) Let $p=1$ and $q=-1$. Example of p & q (A1)

$|p+q|$

$=|1+(-1)|$

$=0$

$|p|+|q|$

$=|1|+|-1|$ Substitution (M1)

$=2$

$\therefore |p+q|<|p|+|q|$

Thus, $|p+q|>|p|+|q|$ is not always true. (AG)

(b) Assume that $4x^3-12x-17=0$ has an integer root $\alpha \in \mathbb{Z}$. Assumption (M1)

$4\alpha^3-12\alpha-17=0$

$4\alpha^3-12\alpha=17$ Group terms of α (M1)

$2(2\alpha^3-6\alpha)=17$ $2(2\alpha^3-6\alpha)$ (A1)

As the left hand side $2(2\alpha^3-6\alpha)$ is even, the right hand side 17 is also even, which contradicts with the fact that 17 is odd. Contradiction (R1)

Thus, the equation $4x^3-12x-17=0$ has no integer roots. (AG)

Exercise 1.12

Let $P(n): \frac{0}{1!}+\frac{1}{2!}+\frac{2}{3!}+\cdots+\frac{n-1}{n!}=1-\frac{1}{n!}$

When $n=1$,

L.H.S.

$=\frac{0}{1!}$

$=0$

R.H.S.

$=1-\frac{1}{1!}$

$=0$

L.H.S. = R.H.S.

Thus, the statement is true when $n=1$. — Case when $n=1$ (R1)

Assume that the statement is true when $n=k$. — Assumption (M1)

$\frac{0}{1!}+\frac{1}{2!}+\frac{2}{3!}+\cdots+\frac{k-1}{k!}=1-\frac{1}{k!}$

When $n=k+1$,

$\frac{0}{1!}+\frac{1}{2!}+\frac{2}{3!}+\cdots+\frac{k-1}{k!}+\frac{(k+1)-1}{(k+1)!}$

$=1-\frac{1}{k!}+\frac{k}{(k+1)!}$ — Apply the assumption (A1)

$=1-\left[\frac{1}{k!}-\frac{k}{(k+1)!}\right]$

$=1-\left[\frac{(k+1)!}{k!(k+1)!}-\frac{k\cdot k!}{k!(k+1)!}\right]$ — Common denominator (A1)

$=1-\frac{(k+1)\cdot k!-k\cdot k!}{k!(k+1)!}$ — $(k+1)!=(k+1)\cdot k!$ (A1)

$=1-\frac{k!(k+1-k)}{k!(k+1)!}$

$=1-\frac{1}{(k+1)!}$

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^{+}$. — (R1)

Exercise 1.13

Let $P(n)$: $3^{2n+1}+1$ is divisible by 4

When $n=1$,

$3^{2(1)+1}+1$

$=27+1$

$=28$

$=4(7)$ which is divisible by 4

Thus, the statement is true when $n=1$. Case when $n=1$ (R1)

Assume that the statement is true when $n=k$. Assumption (M1)

$3^{2k+1}+1=4M$, $M\in\mathbb{Z}$

When $n=k+1$,

$3^{2(k+1)+1}+1$

$=3^{2k+3}+1$

$=3^{2}\cdot 3^{2k+1}+1$ $3^{2k+3}=3^{2}\cdot 3^{2k+1}$ (M1)

$=9(4M-1)+1$ Apply the assumption (A1)

$=36M-9+1$

$=36M-8$

$=4(9M-2)$ which is divisible by 4 $4(9M-2)$ (A1)

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^{+}$. (R1)

Exercise 1.14

(a) (i) $\begin{cases} x+3y-2z=3 \\ -2x-y+z=-1 \\ -x+2y+az=b \end{cases}$

$\rightarrow \begin{cases} x+3y-2z=3 \\ 5y-3z=5 \\ 5y+(a-2)z=b+3 \end{cases}$ $(R_2+2R_1$ $\&R_3+R_1)$ Row operations (M1)(A1)

$\rightarrow \begin{cases} x+3y-2z=3 \\ 5y-3z=5 \\ (a+1)z=b-2 \end{cases}$ (R_3-R_2) $(a+1)z=b-2$ (A1)

Thus, the system has a unique solution when $a+1\neq 0$ and $b-2\in\mathbb{R}$.

$\therefore a\neq -1$ and $b\in\mathbb{R}$ (A1)

(ii) The system has no solutions when $a+1=0$ and $b-2\neq 0$.

$\therefore a=-1$ and $b\neq 2$ (A1)

(iii) The system has infinite number of solutions when $a+1=0$ and $b-2=0$.

$\therefore a=-1$ and $b=2$ (A1)

(b) $\begin{cases} x+3y-2z=3 \\ 5y-3z=5 \\ (3+1)z=6-2 \end{cases}$

$4z=4$

$z=1$ (A1)

$5y-3(1)=5$

$5y=8$

$y=\frac{8}{5}$ (A1)

$x+3\left(\frac{8}{5}\right)-2(1)=3$

$x=\frac{1}{5}$ (A1)

Exercise 1.15

(a) The number of ways

$= 4! \times 6! \times 2$ — $4! \times 6!$ (A1) & two cases (A1)

$= 34560$ — (A1)

(b) The number of ways

$= 4! \times 7!$ — $4!$ (A1) & $7!$ (A1)

$= 120960$ — (A1)

(c) The number of ways

$= P_4^7 \times 6!$ — P_4^7 & $6!$ (A1)

$= 604800$ — (A1)

Exercise 1.16

(a) The number of possible groups

$= C_5^{8+2}$ — C_5^{10} (A1)

$= 252$ — (A1)

(b) (i) The number of groups

$= C_3^8 \times C_2^2$ — C_3^8 or C_2^2 (A1)

$= 56$ — (A1)

(ii) The number of different possible teams

$= C_3^8 \times C_2^2 + C_4^8 \times C_1^2 + C_5^8 \times C_0^2$ — Three cases (A1)

$= 252$ — (A1)

Exercise 1.17

(a) $\alpha+2\mathrm{i}$ and $2-4\mathrm{i}$ are two of the roots

$\therefore \alpha-2\mathrm{i}$ and $2+4\mathrm{i}$ are also two of the roots — Conjugate (A1)

Product of roots $=(-1)^4\dfrac{400}{1}$ — $(-1)^4\dfrac{a_0}{a_4}$ (M1)

$\therefore (\alpha+2\mathrm{i})(\alpha-2\mathrm{i})(2-4\mathrm{i})(2+4\mathrm{i})=(-1)^4\dfrac{400}{1}$ — Correct equation (A1)

$(\alpha^2-2^2\mathrm{i}^2)(2^2-4^2\mathrm{i}^2)=400$

$(\alpha^2+4)(4+16)=400$

$\alpha^2+4=20$

$\alpha^2=16$

$\alpha=-4$ — (A1)

(b) Sum of roots $=-\dfrac{k}{1}$ — $-\dfrac{a_3}{a_4}$ (M1)

$\therefore (-4+2\mathrm{i})+(-4-2\mathrm{i})+(2-4\mathrm{i})+(2+4\mathrm{i})=-\dfrac{k}{1}$ — Correct equation (A1)

$-4=-k$

$k=4$ — (A1)

Exercise 1.18

(a) (i)

$$(\cos\theta + i\sin\theta)^4$$
$$= \cos^4\theta + C_1^4\cos^3\theta(i\sin\theta)^1$$
$$+C_2^4\cos^2\theta(i\sin\theta)^2$$ Binomial theorem (A1)
$$+C_3^4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$$
$$= \cos^4\theta + 4i\cos^3\theta\sin\theta + 6\cos^2\theta(i^2\sin^2\theta)$$
$$+4\cos\theta(i^3\sin^3\theta) + i^4\sin^4\theta$$
$$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta$$
$$-4i\cos\theta\sin^3\theta + \sin^4\theta$$ (A1)

(ii)

$$(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4i\cos^3\theta\sin\theta$$
$$-6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$$
$$\cos 4\theta + i\sin 4\theta = \cos^4\theta + 4i\cos^3\theta\sin\theta$$ De Moivre's theorem (M1)
$$-6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$$
$$\therefore \cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$ Compare real parts (M1)
$$\cos 4\theta = \cos^4\theta$$ $\sin^2\theta = 1 - \cos^2\theta$ (A1)
$$-6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$
$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta$$
$$+1 - 2\cos^2\theta + \cos^4\theta$$
$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$ (AG)

(iii)

$$i\sin 4\theta = 4i\cos^3\theta\sin\theta - 4i\cos\theta\sin^3\theta$$ Compare imaginary parts (M1)
$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$
$$\sin 4\theta = 4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$$
$$\sin 4\theta = 4\sin\theta\cos\theta(1 - \sin^2\theta - \sin^2\theta)$$ $\cos^2\theta = 1 - \sin^2\theta$ (A1)
$$\sin 4\theta = 4\sin\theta\cos\theta(1 - 2\sin^2\theta)$$
$$\sin 4\theta = 4\sin\theta\cos\theta - 8\sin^3\theta\cos\theta$$
$$\therefore \alpha = -8$$ (A1)

(b) $z^4 = \cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{2}\right)$

$z = \cos\left(\frac{\frac{\pi}{2} + 2\pi k}{4}\right) + \mathrm{i}\sin\left(\frac{\frac{\pi}{2} + 2\pi k}{4}\right) \; (k = 0, 1, 2, 3)$ 4 complex roots (M1)

$z = \cos\left(\frac{\pi}{8} + \frac{\pi k}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{8} + \frac{\pi k}{2}\right)$ $\frac{\pi}{8} + \frac{\pi k}{2}$ (A1)

$\therefore r = 1$ (A1)

$\beta = \frac{\pi}{8} + \frac{\pi(0)}{2}$

$\beta = \frac{\pi}{8}$ (A1)

(c) $\sin 4\theta = 4\sin\theta\cos\theta - 8\sin^3\theta\cos\theta$

$\therefore \sin 4\left(\frac{\pi}{8}\right) = 4\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$

$-8\sin^3\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$ $\theta = \frac{\pi}{8}$ (M1)

$\sin\left(\frac{\pi}{2}\right) = 4\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) - 8\sin^3\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$ $4\left(\frac{\pi}{8}\right) = \frac{\pi}{2}$ (A1)

$1 = 4\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) - 8\sin^3\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$ $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ (A1)

$1 = 4\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)\left(1 - 2\sin^2\left(\frac{\pi}{8}\right)\right)$

$4\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)\left(1 - 2\sin^2\left(\frac{\pi}{8}\right)\right) - 1 = 0$ (AG)

(d) (i) $w = R(\cos\lambda + \mathrm{i}\sin\lambda)$ is 2 units below $z = r(\cos\beta + \mathrm{i}\sin\beta)$. (A1)

(ii) $(\mu+2\mathrm{i})^4$

$$=\left(\frac{1-2\mathrm{i}e^{\frac{11\pi}{8}\mathrm{i}}}{e^{\frac{11\pi}{8}\mathrm{i}}}+2\mathrm{i}\right)^4$$ Substitution (M1)

$$=\left(\frac{1}{e^{\frac{11\pi}{8}\mathrm{i}}}-2\mathrm{i}+2\mathrm{i}\right)^4$$

$$=\left(\frac{1}{e^{\frac{11\pi}{8}\mathrm{i}}}\right)^4$$

$$=\frac{1}{e^{\frac{11\pi}{2}\mathrm{i}}}$$ $(e^{\frac{11\pi}{8}\mathrm{i}})^4=e^{\frac{11\pi}{2}\mathrm{i}}$ (A1)

$$=e^{-\frac{11\pi}{2}\mathrm{i}}$$

$$=\cos\left(-\frac{11\pi}{2}\right)+\mathrm{i}\sin\left(-\frac{11\pi}{2}\right)$$

$$=\cos\left(\frac{\pi}{2}\right)+\mathrm{i}\sin\left(\frac{\pi}{2}\right)$$ $-\frac{11\pi}{2}+3(2\pi)=\frac{\pi}{2}$ (A1)

$$=\mathrm{i}$$

Thus, $\mu=\frac{1-2\mathrm{i}e^{\frac{11\pi}{8}\mathrm{i}}}{e^{\frac{11\pi}{8}\mathrm{i}}}$ is a root of $(w+2\mathrm{i})^4=\mathrm{i}$. (AG)

Exercise 1.19

(a) The modulus of wz

$=|w||z|$ — $|wz|=|w||z|$ (M1)

$=\left(\frac{1}{8}\right)\left(\frac{1}{2}\right)$

$=\frac{1}{16}$ — (A1)

(b) z

$=\frac{1}{2}\left(\cos\frac{k\pi}{4}-\mathrm{i}\sin\frac{k\pi}{4}\right)$

$=\frac{1}{2}\left(\cos\left(-\frac{k\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{k\pi}{4}\right)\right)$

$\therefore \arg(z)=-\frac{k\pi}{4}$

The argument of wz

$=\arg(w)+\arg(z)$ — $\arg(wz)=\arg(w)+\arg(z)$ (M1)

$=\frac{\pi}{4}-\frac{k\pi}{4}$

$=\frac{(1-k)\pi}{4}$ — (A1)

(c) $\arg(wz)$

$=\frac{(1-k)\pi}{4}$

$\therefore \frac{(1-k)\pi}{4}=-\frac{\pi}{2},-\frac{3\pi}{2},-\frac{5\pi}{2},\cdots$ — $\frac{\pi}{2}\pm\pi$ (M1)

$1-k=-2,-6,-10,\cdots$ — $-2,-6,-10,\cdots$ (A1)

$-k=-3,-7,-11,\cdots$

$k=3,7,11,\cdots$

Thus, the minimum value of k is 3. — (A1)