Chapter 7 Solution

Exercise 20

1. $\cos 2\alpha = 1 - 2\sin^2 \alpha$

$\cos 2\alpha = 1 - 2\left(\frac{2}{3}\right)^2$ (A1) for substitution

$\cos 2\alpha = \frac{1}{9}$

$\sec 4\alpha = \frac{1}{\cos 2(2\alpha)}$ (M1) for valid approach

$\sec 4\alpha = \frac{1}{2\cos^2 2\alpha - 1}$

$\sec 4\alpha = \frac{1}{2\left(\frac{1}{9}\right)^2 - 1}$ A1

$\sec 4\alpha = -\frac{81}{79}$ A1

[4]

2. $\tan\alpha = \sqrt{\sec^2\alpha - 1}$ (M1) for valid approach

$$\tan\alpha = \sqrt{\frac{1}{\cos^2\alpha} - 1}$$

$$\tan\alpha = \sqrt{\frac{1}{\left(-\frac{3}{5}\right)^2} - 1}$$ (A1) for substitution

$$\tan\alpha = \frac{4}{3}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\tan 2\alpha = \frac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2}$$ A1

$$\tan 2\alpha = \frac{\frac{8}{3}}{-\frac{7}{9}}$$

$$\tan 2\alpha = -\frac{24}{7}$$ A1

[4]

3. $\cos\alpha + \sin\alpha = \dfrac{\sqrt{3}}{2}$

$(\cos\alpha + \sin\alpha)^2 = \left(\dfrac{\sqrt{3}}{2}\right)^2$ (M1) for valid approach

$\cos^2\alpha + 2\sin\alpha\cos\alpha + \sin^2\alpha = \dfrac{3}{4}$

$1 + \sin 2\alpha = \dfrac{3}{4}$ A1

$\sin 2\alpha = -\dfrac{1}{4}$ (A1) for correct value

$\sec 4\alpha = \dfrac{1}{\cos 2(2\alpha)}$ (M1) for valid approach

$\sec 4\alpha = \dfrac{1}{1 - 2\sin^2 2\alpha}$

$\sec 4\alpha = \dfrac{1}{1 - 2\left(-\dfrac{1}{4}\right)^2}$ A1

$\sec 4\alpha = \dfrac{8}{7}$ A1

[6]

4. $(\sec\alpha + \tan\alpha)^2 = \frac{3}{2} + 2\sec\alpha\tan\alpha$

$\sec^2\alpha + 2\sec\alpha\tan\alpha + \tan^2\alpha = \frac{3}{2} + 2\sec\alpha\tan\alpha$ (M1) for valid approach

$\sec^2\alpha + \tan^2\alpha = \frac{3}{2}$

$1 + \tan^2\alpha + \tan^2\alpha = \frac{3}{2}$ A1

$2\tan^2\alpha = \frac{1}{2}$

$\tan^2\alpha = \frac{1}{4}$

$\tan\alpha = \frac{1}{2}$ or $\tan\alpha = -\frac{1}{2}$ (*Rejected*) (A1) for correct value

$\cot 2\alpha = \frac{1 - \tan^2\alpha}{2\tan\alpha}$ (M1) for valid approach

$\cot 2\alpha = \frac{1 - \left(\frac{1}{2}\right)^2}{2\left(\frac{1}{2}\right)}$ A1

$\cot 2\alpha = \frac{3}{4}$ A1

[6]

Exercise 21

1. $\tan 2x = \tan x$

$\frac{2\tan x}{1-\tan^2 x} = \tan x$ (A1) for substitution

$2\tan x = \tan x(1-\tan^2 x)$ (M1) for valid approach

$\tan x(2-(1-\tan^2 x)) = 0$

$\tan x = 0$ or $\tan^2 x = -1$ (*Rejected*) A1

$x = 0$, $x = \pi$, $x = 2\pi$, $x = 3\pi$ or $x = 4\pi$ A2

[5]

2. $\operatorname{cosec}^2 x + 2\cot x = 0$

$\cot^2 x + 1 + 2\cot x = 0$ (A1) for substitution

$\cot^2 x + 2\cot x + 1 = 0$

$(\cot x + 1)^2 = 0$ (A1) for factorization

$\cot x = -1$

$\tan x = -1$ A1

$x = \frac{3\pi}{4}$ A1

[4]

3. $\sec^2 2x + \tan 2x = 1$

$\tan^2 2x + 1 + \tan 2x = 1$ (A1) for substitution

$\tan^2 2x + \tan 2x = 0$

$\tan 2x(\tan 2x + 1) = 0$ (A1) for factorization

$\tan 2x = 0$ or $\tan 2x = -1$ A1

$2x = \pi$, $2x = 2\pi$ or $2x = \pi - \frac{\pi}{4}$, $2x = 2\pi - \frac{\pi}{4}$

$\therefore x = \frac{3\pi}{8}$ (*Rejected*), $x = \frac{\pi}{2}$, $x = \frac{7\pi}{8}$ or $x = \pi$ (*Rejected*) A2

[5]

4. $\tan x + \cot x = 4$

$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = 4$ (A1) for substitution

$\frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = 4$

$1 = 4\sin x \cos x$ (M1) for valid approach

$1 = 2(2\sin x \cos x)$

$\sin 2x = \frac{1}{2}$ A1

$2x = \frac{\pi}{6}$ or $2x = \pi - \frac{\pi}{6}$

$x = \frac{\pi}{12}$ or $x = \frac{11\pi}{12}$ A2

[5]

Exercise 22

1. $\cos\alpha = \sqrt{1-\sin^2\alpha}$

$\cos\alpha = \sqrt{1-\left(\frac{2}{5}\right)^2}$ (A1) for substitution

$\cos\alpha = \frac{\sqrt{21}}{5}$

$\sin\beta = \sqrt{1-\cos^2\beta}$

$\sin\beta = \sqrt{1-\left(\frac{3}{5}\right)^2}$ (A1) for substitution

$\sin\beta = \frac{4}{5}$

$\cos(\alpha-2\beta) = \cos\alpha\cos 2\beta + \sin\alpha\sin 2\beta$ A1

$\cos(\alpha-2\beta) = \cos\alpha(2\cos^2\beta-1) + \sin\alpha(2\sin\beta\cos\beta)$ A1

$\cos(\alpha-2\beta) = \left(\frac{\sqrt{21}}{5}\right)\left(2\left(\frac{3}{5}\right)^2-1\right) + 2\left(\frac{2}{5}\right)\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)$ (A1) for substitution

$\cos(\alpha-2\beta) = \left(\frac{\sqrt{21}}{5}\right)\left(\frac{-7}{25}\right) + \frac{48}{125}$

$\cos(\alpha-2\beta) = \frac{-7\sqrt{21}+48}{125}$ A1

[6]

2. $\sec\alpha = \sqrt{1+\tan^2\alpha}$

$\sec\alpha = \sqrt{1+\left(\frac{3}{4}\right)^2}$ (A1) for substitution

$\sec\alpha = \frac{5}{4}$

$\therefore \cos\alpha = \frac{4}{5}$

$\sin\alpha = \sqrt{1-\cos^2\alpha}$

$\sin\alpha = \sqrt{1-\left(\frac{4}{5}\right)^2}$ (A1) for substitution

$\sin\alpha = \frac{3}{5}$

$\sin\beta = \sqrt{1-\cos^2\beta}$

$\sin\beta = \sqrt{1-\left(\frac{\sqrt{3}}{5}\right)^2}$ (A1) for substitution

$\sin\beta = \frac{\sqrt{22}}{5}$

$\sin(\alpha+2\beta) = \sin\alpha\cos 2\beta + \cos\alpha\sin 2\beta$ A1

$\sin(\alpha+2\beta) = \sin\alpha(2\cos^2\beta - 1) + \cos\alpha(2\sin\beta\cos\beta)$ A1

$\sin(\alpha+2\beta) = \left(\frac{3}{5}\right)\left(2\left(\frac{\sqrt{3}}{5}\right)^2 - 1\right) + 2\left(\frac{4}{5}\right)\left(\frac{\sqrt{22}}{5}\right)\left(\frac{\sqrt{3}}{5}\right)$ (A1) for substitution

$\sin(\alpha+2\beta) = \left(\frac{3}{5}\right)\left(\frac{-19}{25}\right) + \frac{8\sqrt{66}}{125}$

$\sin(\alpha+2\beta) = \frac{-57+8\sqrt{66}}{125}$ A1

[7]

3. $\cos\left(x+\frac{\pi}{6}\right)=\cos\frac{\pi}{3}\cos x$

$\cos x\cos\frac{\pi}{6}-\sin x\sin\frac{\pi}{6}=\cos\frac{\pi}{3}\cos x$ (A1) for substitution

$\frac{\sqrt{3}}{2}\cos x-\frac{1}{2}\sin x=\frac{1}{2}\cos x$ (A1) for correct values

$\sqrt{3}\cos x-\sin x=\cos x$

$\sqrt{3}\cos x-\cos x=\sin x$ A1

$\sin x=(\sqrt{3}-1)\cos x$

$\tan x=\sqrt{3}-1$

$\therefore a=1,\ b=-1$ A2

[5]

4. $\tan\left(x+\frac{\pi}{4}\right)=2$

$\frac{\tan x+\tan\frac{\pi}{4}}{1-\tan x\tan\frac{\pi}{4}}=2$ (A1) for substitution

$\frac{\tan x+1}{1-\tan x}=2$ (A1) for correct value

$\tan x+1=2(1-\tan x)$

$\tan x+1=2-2\tan x$

$3\tan x=1$

$\tan x=\frac{1}{3}$ A1

$\sec^2 x=1+\tan^2 x$ A1

$\sec^2 x=1+\left(\frac{1}{3}\right)^2$

$\sec^2 x=\frac{10}{9}$ A1

[5]

Exercise 23

1. (a) $1 - \tan x \geq 0$ A1

$\tan x \leq 1$

$\tan\left(-\frac{\pi}{2}\right) < \tan x \leq \tan\frac{\pi}{4}$

$-\frac{\pi}{2} < x \leq \frac{\pi}{4}$

Thus, the largest possible domain of f is

$\left\{x : -\frac{\pi}{2} < x \leq \frac{\pi}{4}\right\}$. A1

[2]

(b) $y = \sqrt{1 - \tan x}$

$\Rightarrow x = \sqrt{1 - \tan y}$ (M1) for swapping variables

$x^2 = 1 - \tan y$ M1

$\tan y = 1 - x^2$

$y = \arctan(1 - x^2)$

$\therefore f^{-1}(x) = \arctan(1 - x^2)$ A1

[3]

2. (a) $\arcsin x - \frac{\pi}{6} \geq 0$ A1

$\arcsin x \geq \frac{\pi}{6}$

$\frac{\pi}{6} \leq \arcsin x \leq \frac{\pi}{2}$

$\sin\frac{\pi}{6} \leq x \leq \sin\frac{\pi}{2}$ A1

$\frac{1}{2} \leq x \leq 1$

Thus, the largest possible domain of f is

$\left\{x : \frac{1}{2} \leq x \leq 1\right\}$. A1

[3]

(b) $y = 2\sqrt{\arcsin x - \frac{\pi}{6}}$

$\Rightarrow x = 2\sqrt{\arcsin y - \frac{\pi}{6}}$ (M1) for swapping variables

$\Rightarrow \frac{x}{2} = \sqrt{\arcsin y - \frac{\pi}{6}}$

$\frac{x^2}{4} = \arcsin y - \frac{\pi}{6}$ M1

$\arcsin y = \frac{x^2}{4} + \frac{\pi}{6}$

$y = \sin\left(\frac{x^2}{4} + \frac{\pi}{6}\right)$

$\therefore f^{-1}(x) = \sin\left(\frac{x^2}{4} + \frac{\pi}{6}\right)$ A1

[3]

3. (a) Let $A = \arctan\frac{1}{3}$ and $B = \arctan\frac{1}{7}$.

$A + B = \arctan\frac{1}{m}$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ (M1) for valid approach

$\frac{1}{m} = \frac{\frac{1}{3}+\frac{1}{7}}{1-\left(\frac{1}{3}\right)\left(\frac{1}{7}\right)}$ (A1) for substitution

$\frac{1}{m} = \frac{\frac{10}{21}}{\frac{20}{21}}$

$\frac{1}{m} = \frac{1}{2}$

$\therefore m = 2$ A1

[3]

(b) $\tan(2A+B) = \tan(A+A+B)$

$\tan(2A+B) = \frac{\tan A + \tan(A+B)}{1 - \tan A \tan(A+B)}$ (M1) for valid approach

$\tan(2A+B) = \frac{\frac{1}{3}+\frac{1}{2}}{1-\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)}$ (A1) for substitution

$\tan(2A+B) = \frac{\frac{5}{6}}{\frac{5}{6}}$

$\tan(2A+B) = 1$

$2A + B = \frac{\pi}{4}$

$\therefore 2\arctan\frac{1}{3} + \arctan\frac{1}{7} = \frac{\pi}{4}$ A1

[3]

4. (a) Let $A = \arctan\frac{1}{5}$ and $B = \arctan\frac{1}{7}$.

$A - B = \arctan\frac{1}{r}$

$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$ (M1) for valid approach

$\frac{1}{r} = \frac{\frac{1}{5} - \frac{1}{7}}{1 + \left(\frac{1}{5}\right)\left(\frac{1}{7}\right)}$ (A1) for substitution

$\frac{1}{r} = \frac{\frac{2}{35}}{\frac{36}{35}}$

$\frac{1}{r} = \frac{1}{18}$

$\therefore r = 18$ A1

[3]

(b) $\arctan 5 - \arctan 7 = \left(\frac{\pi}{2} - \arctan\frac{1}{5}\right) - \left(\frac{\pi}{2} - \arctan\frac{1}{7}\right)$ (M1) for valid approach

$\arctan 5 - \arctan 7 = -\left(\arctan\frac{1}{5} - \arctan\frac{1}{7}\right)$

$\arctan 5 - \arctan 7 = -\arctan\frac{1}{18}$ A1

[2]

Exercise 24

1. (a) $f(-x) = f(x)$

$a(-x) + b\cos(-x) = ax + b\cos x$ (M1) for valid approach

$-ax + b\cos x = ax + b\cos x$

$-ax = ax$

$0 = 2ax$

$a = 0$ A1

$b \in \mathbb{R}$ A1

[3]

(b) $h(-x) = \dfrac{f(-x)}{g(-x)}$ M1

$h(-x) = \dfrac{f(x)}{-(-x)^5}$

$h(-x) = -\dfrac{f(x)}{-x^5}$ A1

$h(-x) = -\dfrac{f(x)}{g(x)}$

$h(-x) = -h(x)$

Thus, h is an odd function. AG

[2]

2. (a) $f(-x) = f(x)$

$a\sec(-x) - \dfrac{b}{-x} = a\sec x - \dfrac{b}{x}$ (M1) for valid approach

$a\sec x + \dfrac{b}{x} = a\sec x - \dfrac{b}{x}$

$a \in \mathbb{R}$ A1

$\dfrac{b}{x} = -\dfrac{b}{x}$

$\dfrac{2b}{x} = 0$

$b = 0$ A1

[3]

(b) $h(-x) = f(-x) + g(-x)$ M1

$h(-x) = f(x) + \sin 4(-x)^2$

$h(-x) = f(x) + \sin 4x^2$ A1

$h(-x) = f(x) + g(x)$

$h(-x) = h(x)$

Thus, h is an even function. AG

[2]

3. (a) $f(-x) = \left|\sin\frac{-x}{2}\right|$ M1

$f(-x) = \left|-\sin\frac{x}{2}\right|$ A1

$f(-x) = |-1|\left|\sin\frac{x}{2}\right|$

$f(-x) = \left|\sin\frac{x}{2}\right|$

$f(-x) = f(x)$

Thus, f is an even function. AG

[2]

(b) $h(x) = g(f(x))$

$h(x) = \frac{1}{1+(f(x))^2}$ (M1) for valid approach

$h(-p) = \frac{1}{1+(f(-p))^2}$

$h(-p) = \frac{1}{1+(f(p))^2}$

$h(-p) = h(p)$

$\therefore h(-p) = q$ A1

[2]

4. (a) $f(-x) = (-x)^2 \operatorname{cosec} 2(-x)$ M1

$f(-x) = x^2(-\operatorname{cosec} 2x)$ A1

$f(-x) = -x^2 \operatorname{cosec} 2x$

$f(-x) = -f(x)$

Thus, f is an odd function. AG

[2]

(b) $h(x) = g(f(x))$

$h(x) = \tan f(x)$ (M1) for valid approach

$h(1.1) = \tan f(1.1)$

$h(1.1) = \tan(-f(-1.1))$

$h(1.1) = -\tan f(-1.1)$ A1

$h(1.1) = -g(f(-1.1))$

$h(1.1) = -h(-1.1)$

$\therefore h(1.1) = -r$ A1

[3]

Exercise 25

1. (a) $\frac{2\pi}{B} = 2\left(\frac{3\pi}{4} - \frac{\pi}{4}\right)$

$\frac{2\pi}{B} = \pi$

$B = 2$ A1

$3 = A\text{cosec}2\left(\frac{\pi}{4}\right) + C$

$3 = A + C$

$C = 3 - A$

$\frac{7}{3} = A\text{cosec}2\left(\frac{3\pi}{4}\right) + C$

$\therefore \frac{7}{3} = -A + 3 - A$ (M1) for substitution

$-\frac{2}{3} = -2A$

$A = \frac{1}{3}$ A1

$C = 3 - \frac{1}{3}$

$C = \frac{8}{3}$ A1

[4]

(b) $\left\{y : y \le \frac{7}{3} \text{ or } y \ge 3\right\}$ A1

[1]

2. (a) $B = \frac{\pi}{6} - 0$

$B = \frac{\pi}{6}$ A1

$-1 = A\sec\left(\frac{\pi}{6} - \frac{\pi}{6}\right) + C$

$-1 = A + C$

$C = -1 - A$

$2 = A\sec\left(\frac{\pi}{2} - \frac{\pi}{6}\right) + C$

$\therefore 2 = 2A + (-1 - A)$ (M1) for substitution

$A = 3$ A1

$C = -1 - 3$

$C = -4$ A1

[4]

(b) $\{y : y \le -7 \text{ or } y \ge -1\}$ A1

[1]

3. (a) $\frac{\pi}{B} = \frac{3}{4} - \frac{1}{4}$ (M1) for valid approach

$\frac{\pi}{B} = \frac{1}{2}$

$B = 2\pi$ A1

$0 = A\cot 2\pi\left(\frac{1}{4}\right) + C$

$C = 0$ A1

$\frac{1}{4} = A\cot 2\pi\left(\frac{1}{8}\right) + 0$

$A = \frac{1}{4}$ A1

[4]

(b) $x = \frac{1}{2}$, $x = 1$ and $x = \frac{3}{2}$ A2

[2]

4. (a) $\frac{2\pi}{B}=2\left(\frac{5}{2}-\frac{3}{2}\right)$

$\frac{2\pi}{B}=2$

$B=\pi$ A1

$\pi-4=A\text{cosec}\pi\left(\frac{3}{2}\right)+C$

$\pi-4=-A+C$

$C=\pi-4+A$

$\pi+4=A\text{cosec}\pi\left(\frac{5}{2}\right)+C$

$\therefore \pi+4=A+\pi-4+A$ (M1) for substitution

$8=2A$

$A=4$ A1

$C=\pi-4+4$

$C=\pi$ A1

[4]

(b) $f(x)=8+\pi$

$4\text{cosec}\pi x+\pi=8+\pi$ (A1) for setting equation

$4\text{cosec}\pi x=8$

$\text{cosec}\pi x=2$

$\sin\pi x=\frac{1}{2}$ A1

$\pi x=\frac{\pi}{6}$, $\pi x=\pi-\frac{\pi}{6}$, $\pi x=\frac{\pi}{6}+2\pi$

or $\pi x=\pi-\frac{\pi}{6}+2\pi$

$x=\frac{1}{6}$, $x=\frac{5}{6}$, $x=\frac{13}{6}$ or $x=\frac{17}{6}$ A2

[4]

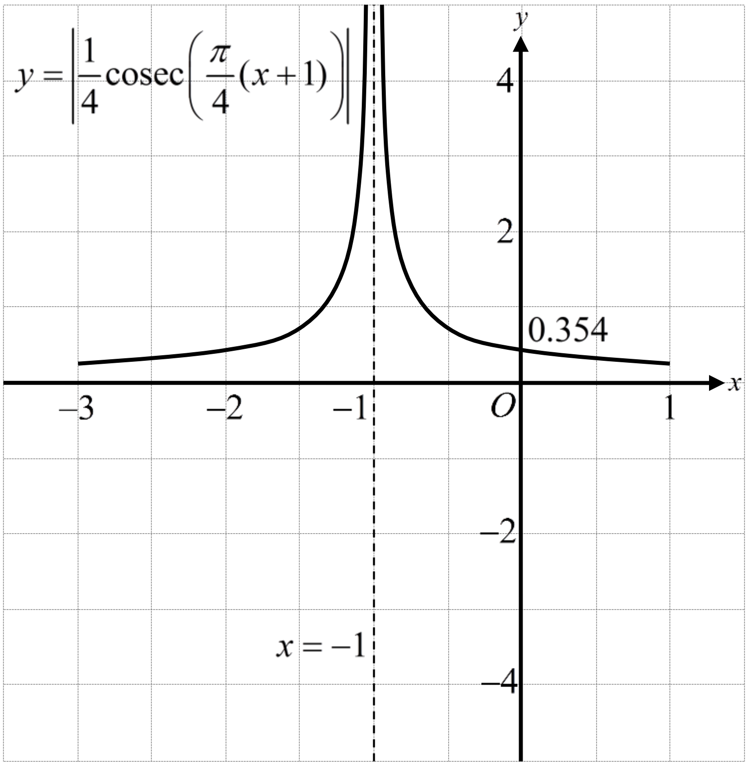
Exercise 26

1. (a) For correct shape A1

For correct asymptote A1

For correct intercept A1

[3]



(b) $\left|\frac{1}{4}\text{cosec}\left(\frac{\pi}{4}(x+1)\right)\right|+x=0$

By considering the graph of

$y=\left|\frac{1}{4}\text{cosec}\left(\frac{\pi}{4}(x+1)\right)\right|+x$, $x=-1.255283$. (M1) for valid approach

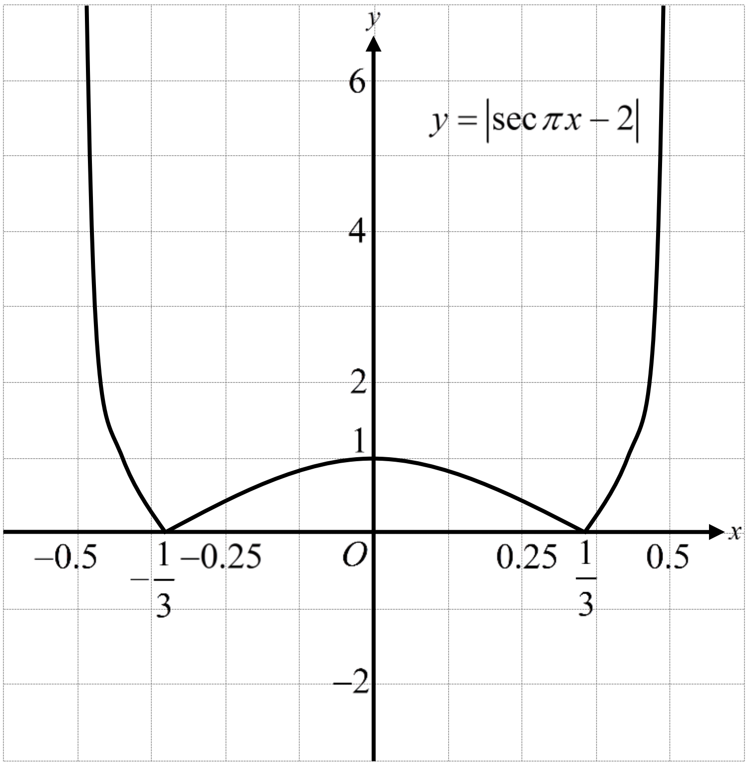
$\therefore x=-1.26$ A1

[2]

2. (a) For correct shape A1

For correct intercepts A2

[3]



(b) $|\sec \pi x - 2| \geq 1$

$|\sec \pi x - 2| - 1 \geq 0$

By considering the graph of $y = |\sec \pi x - 2| - 1$,

$x \leq -0.391826552$, $x = 0$ or $x \geq 0.391826552$. (M1) for valid approach

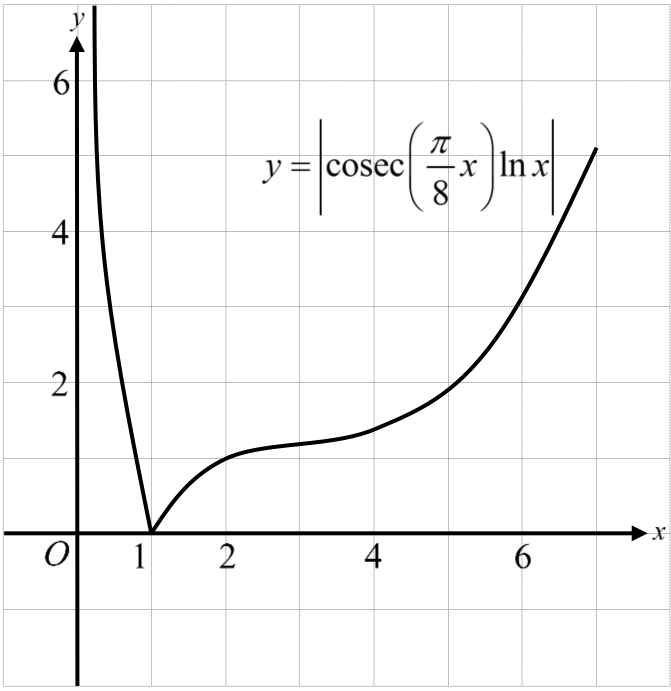
$\therefore -0.5 < x \leq -0.392$, $x = 0$ or $0.392 \leq x < 0.5$ A2

[3]

3. (a) For correct shape — A2

For correct intercept — A1

[3]



(b) $\left|\operatorname{cosec}\left(\frac{\pi}{8}x\right)\ln x\right| + x \geq 4$

$\left|\operatorname{cosec}\left(\frac{\pi}{8}x\right)\ln x\right| + x - 4 \geq 0$

By considering the graph of

$y = \left|\operatorname{cosec}\left(\frac{\pi}{8}x\right)\ln x\right| + x - 4$, $x \leq 0.5032849$ or

$x \geq 2.8380063$. — (M1) for valid approach

$\therefore 0 < x \leq 0.503$ or $2.84 \leq x \leq 7$ — A2

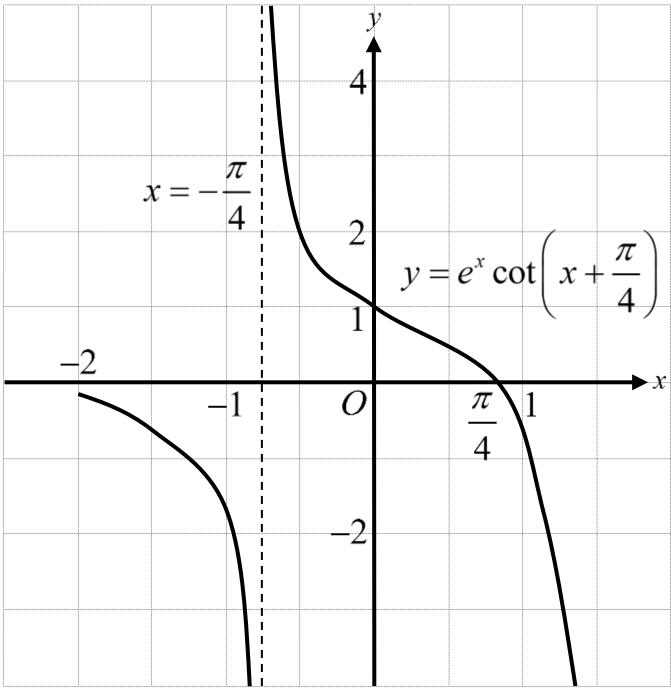
[3]

4. (a) For correct shape A1

For correct asymptote A1

For correct intercepts A1

[3]



(b) $k > -0.0503534375$

$k > -0.0504$ A2

[2]