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Chapter

3

Geometric Sequences

SUMMARY POINTs

✓ $S_\infty = \dfrac{u_1}{1-r}$: The sum to infinity of a geometric sequence u_n, given that $-1 < r < 1$

Solutions of Chapter 3

11 Paper 1 – Sum to Infinity

Example

The first three terms of a geometric sequence are $u_1 = 800$, $u_2 = 720$ and $u_3 = 648$.

(a) Find the value of r.

[2]

(b) Find the value of S_6.

[2]

(c) Find the sum to infinity of this sequence.

[2]

Solution

(a) $r = \frac{720}{800}$ (M1) for valid approach

$r = 0.9$ A1

[2]

(b) $S_6 = \frac{u_1(1-r^6)}{1-r}$

$S_6 = \frac{800(1-(0.9)^6)}{1-0.9}$ (A1) for substitution

$S_6 = 3748.472$

$S_6 = 3750$ A1

[2]

(c) $S_\infty = \frac{u_1}{1-r}$

$S_\infty = \frac{800}{1-0.9}$ (A1) for substitution

$S_\infty = 8000$ A1

[2]

Exercise 11

1. The first three terms of a geometric sequence are $u_1 = -900$, $u_2 = -540$ and $u_3 = -324$.

(a) Find the value of r.

[2]

(b) Find the value of S_{10}.

[2]

(c) Find the sum to infinity of this sequence.

[2]

2. The first three terms of an infinite geometric sequence are $\ln x^{48}$, $\ln x^{24}$ and $\ln x^{12}$, where $x > 0$.

(a) Find the common ratio of the geometric sequence.

[3]

(b) Find u_6.

[2]

(c) Find the sum to infinity of this sequence.

[2]

3. The first three terms of an infinite geometric sequence are e^{12x}, e^{8x} and e^{4x}.

(a) Find the common ratio of the geometric sequence.

[2]

(b) Find u_7.

[2]

(c) Find x if the sum to infinity of this sequence is $\dfrac{e^{96}}{e^{24}-1}$.

[3]

4. The first three terms of an infinite geometric sequence are 3^{10x}, 3^{9x} and 3^{8x}.

(a) Find the common ratio of the geometric sequence.

[2]

(b) Find a general expression for u_n.

[3]

(c) Find the sum to infinity if the common ratio is $\dfrac{1}{3}$, giving the answer in the form $a \times 3^b$.

[3]

12 Paper 1 – Condition of Sum to Infinity

Example

The first three terms of an infinite geometric sequence are $-\frac{6}{r}$, -6, $-6r$, where r is the common ratio. The two possible values of r are $\frac{3}{2}$ and $-\frac{2}{3}$.

(a) State which value of r leads to this sum **and** justify your answer.

[2]

(b) Hence, calculate the sum of the sequence.

[4]

Solution

(a) $r=-\frac{2}{3}$ leads to a finite sum. A1

As $-1<-\frac{2}{3}<1$. R1

[2]

(b) $u_1=-\frac{6}{-\frac{2}{3}}$ (A1) for finding u_1

$u_1=9$ (A1) for correct value

$S_\infty=\frac{u_1}{1-r}$

$S_\infty=\frac{9}{1-\left(-\frac{2}{3}\right)}$ (A1) for substitution

$S_\infty=\frac{27}{5}$ A1

[4]

Exercise 12

1. The first three terms of an infinite geometric sequence are $\frac{10}{r}$, 10, $10r$, where r is the common ratio. The two possible values of r are $\frac{1}{2}$ and -2.

(a) State which value of r leads to this sum **and** justify your answer.

[2]

(b) Hence, calculate the sum of the sequence.

[4]

2. The first three terms of an infinite geometric sequence are $\frac{27}{r^2}$, $\frac{27}{r}$, 27, where r is the common ratio. The two possible values of r are 3 and $-\frac{1}{3}$.

(a) If the sequence has a finite sum, state which value of r leads to this sum **and** justify your answer.

[2]

(b) If the sequence does not have a finite sum, find the sum of the first four terms.

[4]

3. The first three terms of an infinite geometric sequence are $\log_2 x^r$, $\log_2 x^{r^2}$, $\log_2 x^{r^3}$, where r is the common ratio. The two possible values of the common ratio r are $\frac{1}{2}$ and -2.

(a) Consider the value of r such that $-1 < r < 1$. Find S_∞, giving the answer in terms of x.

[3]

(b) Consider the value of r such that $r < -1$. Find S_6 when $x = \frac{1}{2}$.

[4]

4. The first three terms of an infinite geometric sequence are u_1, $u_2 = m + 2$, $u_3 = 9$, where $m \in \mathbb{Z}$. The two possible values of the common ratio r are $\frac{4}{3}$ and $-\frac{1}{3}$.

(a) Consider the value of r such that $-1 < r < 1$. Find m.

[3]

(b) Hence, calculate the sum of the sequence.

[4]

Chapter

6

Complex Numbers

SUMMARY POINTs

✓ Terminologies of complex numbers:

$\mathrm{i}=\sqrt{-1}$: Imaginary unit

$z=a+b\mathrm{i}$: Complex number in Cartesian form

a : Real part of z

b : Imaginary part of z

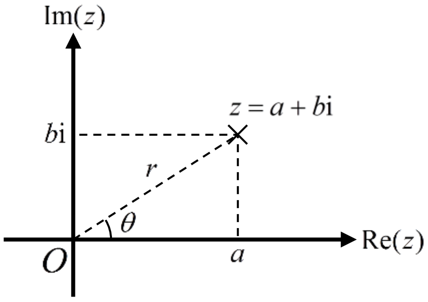
$z^*=a-b\mathrm{i}$: Conjugate of $z=a+b\mathrm{i}$

$|z|=r=\sqrt{a^2+b^2}$: Modulus of $z=a+b\mathrm{i}$

$\arg(z)=\theta=\arctan\frac{b}{a}$: Argument of $z=a+b\mathrm{i}$

✓ Properties of Argand diagram:

1. Real axis: Horizontal axis
2. Imaginary axis: Vertical axis



SUMMARY POINTs

✓ Forms of complex numbers:

1. $z = a + b\mathrm{i}$: Cartesian form
2. $z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{cis}\theta$: Modulus-argument form
3. $z = re^{\mathrm{i}\theta}$: Euler form

✓ Properties of moduli and arguments of complex numbers z_1 and z_2:

1. $|z_1 z_2| = |z_1||z_2|$
2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
3. $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
4. $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

✓ If $z = a + b\mathrm{i}$ is a root of the polynomial equation $p(z) = 0$, then $z^* = a - b\mathrm{i}$ is also a root of $p(z) = 0$

Solutions of Chapter 6

19 Paper 1 – Real and Imaginary Parts

Example

(a) Express $z=\frac{1}{2+\mathrm{i}}$ in the form $a+b\mathrm{i}$, where a, $b\in\mathbb{Q}$.

[2]

(b) Express z^3 in the form $a+b\mathrm{i}$, where a, $b\in\mathbb{Q}$.

[2]

(c) Hence, write down the imaginary part of z^3.

[1]

Solution

(a) $\frac{1}{2+\mathrm{i}}=\frac{2}{5}-\frac{1}{5}\mathrm{i}$ A2

[2]

(b) $z^3=\left(\frac{2}{5}-\frac{1}{5}\mathrm{i}\right)^3$

$z^3=\frac{2}{125}-\frac{11}{125}\mathrm{i}$ A2

[2]

(c) $-\frac{11}{125}$ A1

[1]

Exercise 19

1. (a) Express $z=\frac{1}{3-4\mathrm{i}}$ in the form $a+b\mathrm{i}$, where a, $b\in\mathbb{Q}$.

[2]

(b) Express z^2 in the form $a+b\mathrm{i}$, where a, $b\in\mathbb{Q}$.

[2]

(c) Hence, write down the real part of z^2.

[1]

2. z is a complex number such that $\frac{z}{1-z} = -1-0.5\text{i}$.

(a) Express z in the form $a+b\text{i}$, where a, $b \in \mathbb{Z}$.

[3]

(b) Hence, write down the imaginary part of z.

[1]

3. z is a complex number such that $2z-1-\text{i} = 5+7\text{i}$.

(a) Express z in the form $a+b\text{i}$, where a, $b \in \mathbb{Z}$.

[2]

(b) Express z^4 in the form $a+b\text{i}$, where a, $b \in \mathbb{Z}$.

[2]

(c) Hence, write down the real part of z^4.

[1]

4. z is a complex number such that $\frac{z}{5-12\text{i}} = \frac{24-7\text{i}}{\text{i}}$.

(a) Express z in the form $a+b\text{i}$, where a, $b \in \mathbb{Z}$.

[2]

(b) Express $(\text{i}^3 z)^2$ in the form $a+b\text{i}$, where a, $b \in \mathbb{Z}$.

[2]

(c) Hence, write down the imaginary part of $(\text{i}^3 z)^2$.

[1]

20 Paper 1 – Moduli and Arguments

Example

Consider the complex number $z=\frac{5i}{3+4i}$, where $z\in\mathbb{C}$.

(a) Express z in the form $a+ib$, where a, $b\in\mathbb{Q}$.

[2]

(b) Find the exact value of the modulus of z.

[2]

(c) Find the value of the argument of z.

[2]

Solution

(a) $z=\frac{5i}{3+4i}$

$z=\frac{4}{5}+\frac{3}{5}i$ A2

[2]

(b) The modulus of z

$=\sqrt{\left(\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2}$ M1

$=\sqrt{1}$

$=1$ A1

[2]

(c) The argument of z

$=\tan^{-1}\left(\frac{\frac{3}{5}}{\frac{4}{5}}\right)$ M1

$=\tan^{-1}\left(\frac{3}{4}\right)$

$=0.6435011088\text{ rad}$

$=0.644\text{ rad}$ A1

[2]

Exercise 20

1. Consider the complex number $z=\frac{2-\mathrm{i}}{2+\mathrm{i}}$, where $z\in\mathbb{C}$.

(a) Express z in the form $a+\mathrm{i}b$, where a, $b\in\mathbb{Q}$.

[2]

(b) Find the exact value of the modulus of z.

[2]

(c) Find the value of the argument of z.

[2]

2. Consider the complex number $z=\frac{10}{13}-\frac{24}{13}\mathrm{i}$, where $z\in\mathbb{C}$.

(a) Express z^2 in the form $a+\mathrm{i}b$, where a, $b\in\mathbb{Q}$.

[2]

(b) Find the exact value of the modulus of z^2.

[2]

(c) Find the value of the argument of z^2.

[2]

3. Consider the complex number $z=\frac{8}{5}-\frac{6}{5}\mathrm{i}$, where $z\in\mathbb{C}$.

(a) (i) Express z^3 in the form $a+\mathrm{i}b$, where a, $b\in\mathbb{Q}$.

(ii) Hence, write down $(z^3)^*+\frac{352}{125}$ in the form $a+\mathrm{i}b$, where a, $b\in\mathbb{Q}$.

[3]

(b) Find the exact value of the modulus of $(z^3)^*+\frac{352}{125}$.

[2]

(c) Write down the exact value of the argument of $(z^3)^*+\frac{352}{125}$.

[1]

4. The complex numbers z_1 and z_2 have arguments between 0 and π radians. Given that $z_1+\mathrm{i}z_2=0$, $z_2{}^2=-2-2\sqrt{a}\mathrm{i}$ and $|z_2|=2$, where $a\in\mathbb{R}$.

(a) Find the modulus of z_1.

[2]

(b) Find the value of a.

[3]

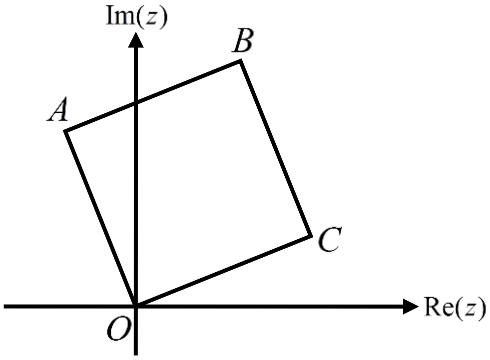
(c) Hence, find the argument of $z_2{}^2$.

[2]

21 Paper 1 – Argand Diagrams

Example

In the following Argand diagram with the origin O, the point A represents the complex number $-4+10i$. The shape of OABC is a square.



(a) Determine the complex numbers represented by

(i) the point B;

(ii) the point C.

[4]

(b) Hence, find the area of OABC.

[2]

Solution

(a) (i) $z_B = -4+10i+(10+4i)$ M1

$z_B = 6+14i$ A1

$z_C = 0+0i+(10+4i)$ M1

$z_C = 10+4i$ A1

[4]

(b) The area of OABC

$= (OA)^2$ M1

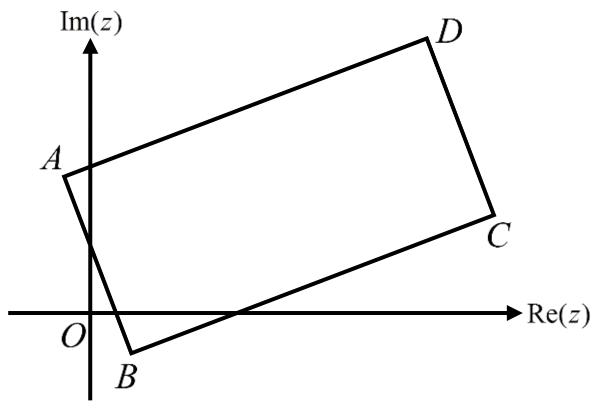
$= (\sqrt{(-4)^2+10^2})^2$

$= 116$ A1

[2]

Exercise 21

1. In the following Argand diagram with the origin O, the point A and the point B represent the complex numbers $-2+9\text{i}$ and $3-3\text{i}$ respectively. The shape of $ABCD$ is a rectangle such that $AD = 2AB$.



(a) Write down

(i) $\text{Re}(3-3\text{i}) - \text{Re}(-2+9\text{i})$;

(ii) $\text{Im}(3-3\text{i}) - \text{Im}(-2+9\text{i})$.

[2]

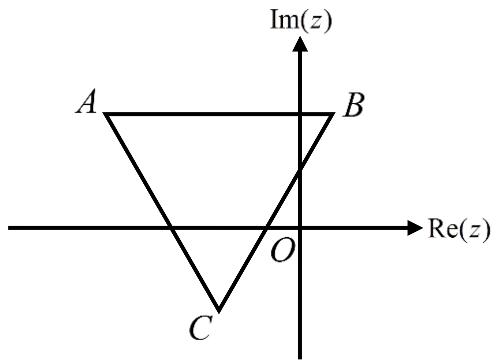
(b) Find the length of AB.

[2]

(c) Hence, find the area of $ABCD$.

[2]

2. In the following Argand diagram with the origin O, the point A represents the complex numbers $-18+10\text{i}$. The shape of ABC is an equilateral triangle with the horizontal side $AB = 20$.



(a) Determine the complex numbers represented by

(i) the point B;

(ii) the point C, giving the answer in exact value.

[4]

(b) Find the area of ABC.

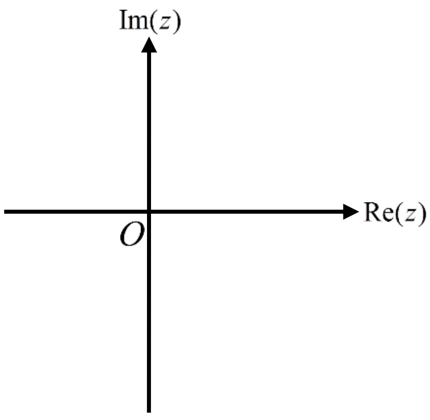
[2]

6

3. In an Argand diagram with the origin O, the points A, B, C and D represent the complex numbers $z = 4 + 4\text{i}$, z^*, $\omega = -2 + 2\text{i}$ and ω^* respectively.

(a) Sketch the points A, B, C and D on the following Argand diagram, and sketch the quadrilateral ABDC.

[3]



(b) Find $\arg(\omega)$.

[2]

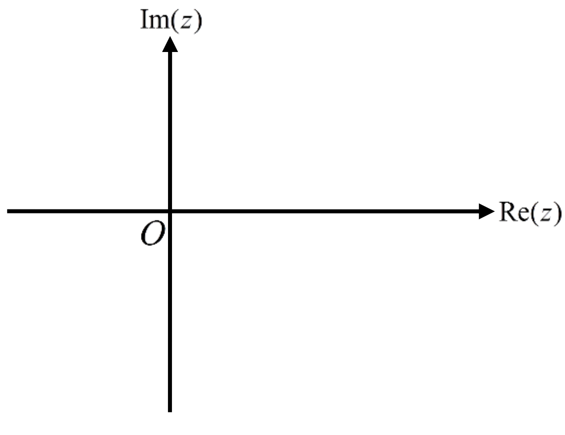
(c) Find the area of the quadrilateral ABDC.

[2]

4. In an Argand diagram with the origin O, the points A, B and C represent the complex numbers $z = -3 + 6\text{i}$, $z - 6 - 15\text{i}$ and $(18 + 9\text{i})^*$ respectively.

(a) Sketch the points A, B and C on the following Argand diagram, and sketch the triangle ABC.

[3]



(b) Find $\arg(z - 6 - 15\text{i})$.

[2]

(c) Find the exact area of the triangle ABC.

[2]

22 Paper 1 – Forms of Complex Numbers

Example

Consider the complex numbers $z_1 = \text{cis}\frac{\pi}{6}$ and $z_2 = 6\text{cis}\frac{2\pi}{3}$.

(a) Express $z_1 z_2$ in the form

(i) $r\text{cis}\theta$;

(ii) $re^{i\theta}$.

[3]

(b) Hence, find the imaginary part of $z_1 z_2$.

[2]

Solution

(a) (i) $z_1 z_2 = \left(\text{cis}\frac{\pi}{6}\right)\left(6\text{cis}\frac{2\pi}{3}\right)$

$z_1 z_2 = (1)(6)\text{cis}\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)$ (M1) for valid approach

$z_1 z_2 = 6\text{cis}\frac{5\pi}{6}$ A1

(ii) $z_1 z_2 = 6e^{\frac{5\pi}{6}i}$ A1

[3]

(b) The imaginary part of $z_1 z_2$

$= 6\sin\frac{5\pi}{6}$ (M1) for valid approach

$= 3$ A1

[2]

6

Exercise 22

1. Consider the complex numbers $z_1 = 12\text{cis}\frac{7\pi}{6}$ and $z_2 = 4\text{cis}\frac{\pi}{2}$.

(a) Express $\frac{z_1}{z_2}$ in the form

(i) $r\text{cis}\theta$;

(ii) $re^{i\theta}$.

[3]

(b) Hence, find the real part of $\frac{z_1}{z_2}$.

[2]

2. Consider the complex numbers $z_1 = 18\sqrt{3}\text{cis}\left(-\frac{\pi}{6}\right)$ and $z_2 = \frac{1}{9}\text{cis}\frac{\pi}{3}$.

(a) Express z_1z_2 in the form

(i) $r\text{cis}\theta$;

(ii) $re^{i\theta}$.

[3]

(b) Hence, find the real part of z_1z_2.

[2]

3. Consider the complex numbers $z_1 = 2\text{cis}\frac{\pi}{12}$ and $z_2 = 3\text{cis}\frac{\pi}{4}$.

(a) (i) Express z_1^2 in the form $r\text{cis}\theta$.

(ii) Hence, find the imaginary part of z_1^2.

[4]

(b) Express $z_1^2z_2$ in the form

(i) $r\text{cis}\theta$;

(ii) $re^{i\theta}$.

[3]

4. Consider the complex numbers $z_1 = \frac{1}{3}\text{cis}\frac{\pi}{6}$ and $z_2 = \frac{1}{9}\text{cis}\frac{11\pi}{12}$.

(a) (i) Express $z_1{}^4$ in the form $r\text{cis}\theta$.

(ii) Hence, find the real part of $z_1{}^4$.

[4]

(b) Express $\frac{z_2}{z_1{}^4}$ in the form

(i) $r\text{cis}\theta$;

(ii) $r\text{e}^{\text{i}\theta}$.

[3]

6

23 Paper 1 – Complex Roots of Quadratic Equations

Example

A quadratic function is given by $f(x)=x^2-6x+58$. It is given that the range of $f(x)$ is $\{y: y\geq 49\}$.

(a) Explain why there is no real root for the equation $f(x)=0$.

[1]

(b) Find the complex roots of the equation $f(x)=0$, giving the answer in the form $a+b\text{i}$, where a, $b\in\mathbb{Q}$.

[3]

(c) If the above two complex roots are located on an Argand diagram, write down the distance between the roots.

[1]

Solution

(a) The range of $f(x)$ is $\{y: y\geq 49\}$, means the graph of $f(x)$ does not have any x-intercept. R1

[1]

(b) $x^2-6x+58=0$

$x=\dfrac{-(-6)\pm\sqrt{(-6)^2-4(1)(58)}}{2(1)}$ (A1) for substitution

$x=\dfrac{6\pm\sqrt{-196}}{2}$ (A1) for simplification

$x=\dfrac{6\pm\sqrt{196}\text{i}}{2}$

$x=3\pm 7\text{i}$ A1

[3]

(c) 14 A1

[1]

Exercise 23

1. A quadratic function is given by $f(x) = -x^2 + 4x - 29$. It is given that the range of $f(x)$ is $\{y : y \le -25\}$.

(a) Explain why there is no real root for the equation $f(x) = 0$.

[1]

(b) Find the complex roots of the equation $f(x) = 0$, giving the answer in the form $a + b\mathrm{i}$, where a, $b \in \mathbb{Q}$.

[3]

(c) If the above two complex roots are located on an Argand diagram, write down the distance between the roots.

[1]

2. A quadratic function is given by $f(x) = (x+5)^2 + 64$.

(a) Explain why there is no real root for the equation $f(x) = 0$.

[1]

(b) Find the complex roots of the equation $f(x) = 0$, giving the answer in the form $a + b\mathrm{i}$, where a, $b \in \mathbb{Q}$.

[3]

(c) If the above two complex roots are located on an Argand diagram, write down the distance between the roots.

[1]

3. A quadratic function is given by $f(x) = ax^2 + bx + c$. It is given that the complex roots of $f(x) = 0$ are $4 + 13\mathrm{i}$ and $4 - 13\mathrm{i}$.

(a) Write down the values of

(i) $(4+13\mathrm{i}) + (4-13\mathrm{i})$;

(ii) $(4+13\mathrm{i})(4-13\mathrm{i})$.

[2]

(b) Hence, find the expression of $f(x)$, giving the answer in terms of a.

[3]

The graph of $f(x)$ passes through $(4, 169)$.

(c) Find the value of a.

[2]

4. A quadratic function is given by $f(x)=ax^2+bx+c$. It is given that the complex roots of $f(x)=0$ are $-\frac{1}{2}+2\text{i}$ and $-\frac{1}{2}-2\text{i}$.

(a) Write down the values of

(i) $\left(-\frac{1}{2}+2\text{i}\right)+\left(-\frac{1}{2}-2\text{i}\right)$;

(ii) $\left(-\frac{1}{2}+2\text{i}\right)\left(-\frac{1}{2}-2\text{i}\right)$.

[2]

(b) Hence, find the expression of $f(x)$, giving the answer in terms of a.

[3]

The graph of $f(x)$ passes through $(0,-17)$.

(c) Find the value of a.

[2]

24 Paper 2 – Applications of Complex Numbers

Example

Two alternating current electrical sources are given as $V_1 = 4\cos(4t + 0.1)$ and $V_2 = 3\cos 4t$ respectively, where t represents time in seconds. The total voltage V is given by $V = V_1 + V_2$.

(a) Write down the amplitude of

(i) V_1;

(ii) V_2.

[2]

(b) Find the period of V_2.

[2]

It is given that $V_1 + V_2 = \text{Re}(e^{4ti}(z + w))$, z, $w \in \mathbb{C}$.

(c) Find the expression of $z + w$.

[3]

(d) Express the following in the form $r(\cos\theta + i\sin\theta)$:

(i) z

(ii) w

[2]

(e) It is given that $z + w = Le^{i\alpha}$. Find

(i) L;

(ii) α.

[6]

(f) Using $V_1 + V_2 = \text{Re}(e^{4ti}(z + w))$, express V in the form $A\cos(Bt + C)$, A, B, $C \in \mathbb{R}$.

[3]

(g) Hence, find the total voltage when $t = 1$.

[2]

Solution

(a) (i) 4 A1

(ii) 3 A1

[2]

(b) The period of V_2

$=\frac{2\pi}{4}$ (M1) for valid approach

$=\frac{\pi}{2}$ s A1

[2]

(c) $V_1+V_2=4\cos(4t+0.1)+3\cos 4t$

$V_1+V_2=\text{Re}(4e^{(4t+0.1)\text{i}})+\text{Re}(3e^{4t\text{i}})$ (M1) for valid approach

$V_1+V_2=\text{Re}(4e^{(4t+0.1)\text{i}}+3e^{4t\text{i}})$ (A1) for correct approach

$V_1+V_2=\text{Re}(e^{4t\text{i}}(4e^{0.1\text{i}}+3))$

$\therefore z+w=4e^{0.1\text{i}}+3$ A1

[3]

(d) (i) $z=4e^{0.1\text{i}}$

$z=4(\cos 0.1+\text{i}\sin 0.1)$ A1

(ii) $w=3$

$w=3(\cos 0+\text{i}\sin 0)$ A1

[2]

(e) (i) $z+w=4(\cos 0.1+\text{i}\sin 0.1)+3(\cos 0+\text{i}\sin 0)$

$z+w=(4\cos 0.1+3\cos 0)$

$+\text{i}(4\sin 0.1+3\sin 0)$ (M1) for valid approach

$z+w=6.980016661+0.3993336666\text{i}$ (A1) for correct values

$L=\sqrt{6.980016661^2+0.3993336666^2}$ M1

$L=6.991430466$

$L=6.99$ A1

(ii) $\alpha=\tan^{-1}\frac{0.3993336666}{6.980016661}$ M1

$\alpha=0.0571486937$

$\alpha=0.0571$ A1

[6]

(f) $V_1 + V_2 = \text{Re}(e^{4ti}(z+w))$

$V_1 + V_2 = \text{Re}(e^{4ti} \cdot 6.991430466e^{0.0571486937i})$ (M1) for substitution

$V_1 + V_2 = \text{Re}(6.991430466e^{4ti+0.0571486937i})$ (A1) for correct approach

$V_1 + V_2 = 6.991430466\cos(4t + 0.0571486937)$

$V_1 + V_2 = 6.99\cos(4t + 0.0571)$ A1

[3]

(g) The required total voltage

$= 6.991430466\cos(4(1) + 0.0571486937)$ (M1) for substitution

$= -4.260226648$ V

$= -4.26$ V A1

[2]

6

Exercise 24

1. Two sound waves are given as $S_1 = 2\sin(6t - 0.1)$ and $S_2 = 3\sin(6t + 0.25)$ respectively, where S_1 and S_2 represent the amplitudes of the two sound waves respectively, in millimetres, and t represents time in seconds. The total amplitude S is given by $S = S_1 + S_2$.

(a) Write down the amplitude of

(i) S_1;

(ii) S_2.

[2]

(b) Find the period of S_2.

[2]

It is given that $S_1 + S_2 = \text{Im}(e^{6ti}(z+w))$, z, $w \in \mathbb{C}$.

(c) Find the expression of $z + w$.

[3]

(d) Express the following in the form $r(\cos\theta + i\sin\theta)$:

(i) z

(ii) w

[2]

(e) It is given that $z+w=Le^{\mathrm{i}\alpha}$. Find

(i) L;

(ii) α.

[6]

(f) Using $S_1+S_2=\mathrm{Im}(e^{6t\mathrm{i}}(z+w))$, express S in the form $A\sin(Bt+C)$, A, B, $C\in\mathbb{R}$.

[3]

(g) Hence, write down the minimum total amplitude.

[1]

2. Two waves are given as $W_1=5\cos(\pi t-0.9)$ and $W_2=7\cos(\pi t-1.3)$ respectively, where W_1 and W_2 represent the amplitudes of the two waves respectively. t represents time in seconds. The total amplitude W is given by $W=W_1+W_2$.

(a) Write down the amplitude of

(i) W_1;

(ii) W_2.

[2]

(b) Find the period of W_2.

[2]

It is given that $W_1+W_2=\mathrm{Re}(e^{\pi t\mathrm{i}}(z+w))$, z, $w\in\mathbb{C}$.

(c) Find the expression of $z+w$.

[3]

(d) Express the following in the form $r(\cos\theta+i\sin\theta)$:

(i) z

(ii) w

[2]

(e) It is given that $z+w=Le^{\mathrm{i}\alpha}$. Find

(i) L;

(ii) α.

[6]

(f) Using $W_1+W_2=\mathrm{Re}(e^{\pi t\mathrm{i}}(z+w))$, express W in the form $A\cos(Bt+C)$, A, B, $C\in\mathbb{R}$.

[3]

(g) Hence, find t when $W=0$, $1<t<2$.

[2]

3. Two sound waves are given as $S_1 = 8\cos(10t + 0.05)$ and S_2 respectively, where S_1 and S_2 represent the amplitudes of the two sound waves respectively, in millimetres, and t represents time in seconds. The total amplitude S is given by
$S = S_1 + S_2 = 10\cos(10t + 0.15)$.

(a) For S, write down its

(i) amplitude;

(ii) period.

[2]

It is given that $S_2 = \text{Re}(e^{10ti}(z - w))$, z, $w \in \mathbb{C}$.

(b) Find the expression of $z - w$.

[3]

(c) Express the following in the form $r(\cos\theta + i\sin\theta)$:

(i) z

(ii) w

[2]

(d) It is given that $z - w = Le^{i\alpha}$. Find

(i) L;

(ii) α.

[6]

(e) Express S_2 in the form $A\cos(Bt + C)$, A, B, $C \in \mathbb{R}$.

[3]

(f) Hence, find the value of t when $S_2 = 1.5$, $9.5 < t < 10$.

[2]

6

4. Two alternating current electrical sources are given as $V_1 = 7\sin(2\pi t - 0.95)$ and V_2 respectively, where t represents time in seconds. The total voltage V is given by $V = V_1 + V_2 = 6.3\sin(2\pi t - 0.5)$.

(a) For V_1, write down its

(i) amplitude;

(ii) period.

[2]

It is given that $V_2 = \text{Im}(e^{2\pi t\text{i}}(z - w))$, z, $w \in \mathbb{C}$.

(b) Find the expression of $z - w$.

[3]

(c) Express the following in the form $r(\cos\theta + i\sin\theta)$:

(i) z

(ii) w

[2]

(d) It is given that $z - w = Le^{\text{i}\alpha}$. Find

(i) L;

(ii) α.

[6]

(e) Express V_2 in the form $A\sin(Bt + C)$, A, B, $C \in \mathbb{R}$.

[3]

(f) Hence, find the range of values of t when $V_2 > 2$, $0.5 \le t \le 1.5$.

[3]

Chapter

7

Matrices

7

SUMMARY POINTs

✓ Terminologies of matrices:

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$: A $m\times n$ matrix with m rows and n columns

a_{ij} : Element on the i th row and the j th column

$$\mathbf{I}=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$: Identity matrix

$$\mathbf{0}=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$: Zero matrix

SUMMARY POINTs

✓ Terminologies of matrices:

$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$: Diagonal matrix

$|\mathbf{A}| = \det(\mathbf{A})$: Determinant of $\mathbf{A}$

$\mathbf{A}$ is non-singular if $\det(\mathbf{A}) \neq 0$

$\mathbf{A}^{-1}$: Inverse of $\mathbf{A}$

$\mathbf{A}^{-1}$ exists if $\mathbf{A}$ is non-singular

✓ For any 2×2 square matrices $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

1. $|\mathbf{A}| = \det(\mathbf{A}) = ad - bc$: Determinant of $\mathbf{A}$
2. $\mathbf{A}^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$: Inverse of $\mathbf{A}$

✓ Operations of matrices:

1. $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \pm \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} \pm b_{11} & \cdots & a_{1n} \pm b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & \cdots & a_{mn} \pm b_{mn} \end{pmatrix}$
2. $k\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & \ddots & \vdots \\ ka_{m1} & \cdots & ka_{mn} \end{pmatrix}$
3. $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$: The element on the i th row and the j th column of $\mathbf{C} = \mathbf{AB} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nk} \end{pmatrix}$, where $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ and $n \times k$ matrices respectively

SUMMARY POINTs

✓ A 2×2 system $\begin{cases} ax+by=c \\ dx+ey=f \end{cases}$ can be expressed as $\mathbf{AX}=\mathbf{B}$, where $\mathbf{X}=\begin{pmatrix} x \\ y \end{pmatrix}$ can be solved by $\mathbf{X}=\mathbf{A}^{-1}\mathbf{B}$

✓ A 3×3 system $\begin{cases} ax+by+cz=d \\ ex+fy+gz=h \\ ix+jy+kz=l \end{cases}$ can be expressed as $\mathbf{AX}=\mathbf{B}$, where $\mathbf{X}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ can be solved by $\mathbf{X}=\mathbf{A}^{-1}\mathbf{B}$

✓ Eigenvalues and eigenvectors of $\mathbf{A}$:

1. $\det(\mathbf{A}-\lambda\mathbf{I})$: Characteristic polynomial of $\mathbf{A}$
2. Solution(s) of $\det(\mathbf{A}-\lambda\mathbf{I})=0$: Eigenvalue(s) of $\mathbf{A}$
3. $\mathbf{v}$: Eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, which satisfies $\mathbf{Av}=\lambda\mathbf{v}$

✓ Diagonalization of $\mathbf{A}$:

1. $\mathbf{D}=\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$: Diagonal matrix of the eigenvalues of $\mathbf{A}$
2. $\mathbf{V}=\left(\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n\right)$: A matrix of the eigenvectors of $\mathbf{A}$
3. $\mathbf{A}=\mathbf{VDV}^{-1} \Rightarrow \mathbf{A}^n=\mathbf{VD}^n\mathbf{V}^{-1}$

SUMMARY POINTs

✓ Two-dimensional transformation matrices:

1. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$: Reflection about the x-axis
2. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$: Reflection about the y-axis
3. $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$: Reflection about the line $y = mx$, where $m = \tan\theta$
4. $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$: Vertical stretch with scale factor k
5. $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$: Horizontal stretch with scale factor k
6. $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$: Enlargement about the origin with scale factor k
7. $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$: Rotation with positive angle θ anticlockwise about the origin
8. $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$: Rotation with positive angle θ clockwise about the origin

Solutions of Chapter 7

25 Paper 1 – Matrix Algebra

Example

Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \\ -2 & 2 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3x^2 & 4x^2 \\ 2x^3 & 3x^3 \end{pmatrix}$.

(a) Write down $\det \mathbf{A}$.

[1]

(b) Find $\det \mathbf{B}$.

[2]

(c) Solve the equation $4\det \mathbf{A} = \det \mathbf{B}$.

[2]

Solution

(a) 8 — A1

[1]

(b) $\det \mathbf{B} = (3x^2)(3x^3) - (4x^2)(2x^3)$ — (A1) for substitution

$\det \mathbf{B} = 9x^5 - 8x^5$

$\det \mathbf{B} = x^5$ — A1

[2]

(c) $4\det \mathbf{A} = \det \mathbf{B}$

$\therefore 4(8) = x^5$ — (M1) for setting equation

$x^5 = 32$

$x = 2$ — A1

[2]

Exercise 25

1. Let $\mathbf{A} = \begin{pmatrix} 2 & 4 & 1 \\ -3 & -5 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2x^2 & 4 \\ x & x \end{pmatrix}$.

(a) Write down $\det \mathbf{A}$.

[1]

(b) Find $\det \mathbf{B}$.

[2]

(c) Solve the equation $(11 - \det \mathbf{A})x = \det \mathbf{B}$.

[2]

2. Let $\mathbf{A} = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & -5 \\ 1 & 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1+x & x \\ -2x & 1-x \end{pmatrix}$.

(a) Write down $\det \mathbf{A}$.

[1]

(b) Find $\det \mathbf{B}$.

[2]

(c) Solve the equation $\det \mathbf{A} + \det \mathbf{B} - 5x = 0$.

[3]

3. Let $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 3e^x & e^{2x} \end{pmatrix}$.

(a) Find $\det \mathbf{A}$.

[2]

(b) Solve the equation $\det \mathbf{A} - 1 = 0$.

[3]

4. Let $\mathbf{A} = \begin{pmatrix} \ln x & 3 \\ -2 & \ln x \end{pmatrix}$.

(a) Find $\det \mathbf{A}$.

[2]

(b) Solve the equation $\det \mathbf{A} = 5\ln x$, giving the answer(s) in terms of e.

[4]

26 Paper 1 – Inverse of a Matrix

Example

Let $\mathbf{A} = \begin{pmatrix} 3 & 4 & 5 \\ 2 & -3 & -4 \\ 2 & -1 & 0 \end{pmatrix}$.

(a) Write down $\mathbf{A}^{-1}$.

[2]

It is given that $\mathbf{AB} + \mathbf{I} = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & -1 \\ 3 & -3 & 2 \end{pmatrix}$, where $\mathbf{B}$ and $\mathbf{I}$ are 3×3 matrices.

(b) Find $\mathbf{B}$.

[3]

7

Solution

(a) $\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{6} & \frac{5}{24} & \frac{1}{24} \\ \frac{1}{3} & \frac{5}{12} & -\frac{11}{12} \\ -\frac{1}{6} & -\frac{11}{24} & \frac{17}{24} \end{pmatrix}$ A2

[2]

(b) $\mathbf{AB} + \mathbf{I} = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & -1 \\ 3 & -3 & 2 \end{pmatrix}$

$\mathbf{AB} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 0 & -1 \\ 3 & -3 & 1 \end{pmatrix}$

$\mathbf{B} = \mathbf{A}^{-1} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 0 & -1 \\ 3 & -3 & 1 \end{pmatrix}$ (M1) for valid approach

$$\mathbf{B}=\begin{pmatrix}\frac{7}{24} & \frac{3}{8} & 0\\ -\frac{29}{12} & \frac{15}{4} & -1\\ \frac{47}{24} & -\frac{21}{8} & 1\end{pmatrix}$$ A2

[3]

Exercise 26

1. Let $\mathbf{A}=\begin{pmatrix}-6 & -3 & 1\\ 1 & 4 & -2\\ 1 & -2 & 1\end{pmatrix}$.

(a) Write down $\mathbf{A}^{-1}$.

[2]

It is given that $\mathbf{AB}+\begin{pmatrix}1 & -2 & -2\\ 0 & 1 & 3\\ -1 & 0 & -3\end{pmatrix}=2\mathbf{I}$, where $\mathbf{B}$ and $\mathbf{I}$ are 3×3 matrices.

(b) Find $\mathbf{B}$.

[3]

2. Let $\mathbf{A}=\begin{pmatrix}3 & 1 & 0\\ 2 & 1 & 1\\ -1 & 1 & -1\end{pmatrix}$.

(a) Write down $\mathbf{A}^{-1}$.

[2]

It is given that $\mathbf{AB}=\begin{pmatrix}-8 & 5 & 3\\ 2 & 6 & 7\\ 5 & -4 & -4\end{pmatrix}-5\mathbf{I}$, where $\mathbf{B}$ and $\mathbf{I}$ are 3×3 matrices.

(b) Find $\mathbf{B}$.

[3]

3. Let $\mathbf{A}=\begin{pmatrix} 2 & 3 & 2 \\ -3 & 3 & -4 \\ 2 & 1 & 2 \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} 10 & -7 & 2 \\ 5 & -9 & 6 \\ -4 & 3 & 8 \end{pmatrix}$.

(a) Write down $\mathbf{A}^{-1}$.

[2]

It is given that $\mathbf{A}^{-1}\mathbf{C}\mathbf{A}=\frac{1}{2}\mathbf{B}$, where $\mathbf{C}$ is a 3×3 matrix.

(b) Find $\mathbf{C}$.

[3]

4. Let $\mathbf{A}=\begin{pmatrix} 0 & 4 & 5 \\ 0 & 2 & 3 \\ 1 & -5 & 7 \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} 1 & 2 & -1 \\ 3 & -3 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Write down $\mathbf{A}^{-1}$.

[2]

It is given that $\mathbf{A}\mathbf{C}\mathbf{A}^{-1}=\mathbf{B}^{3}$, where $\mathbf{C}$ is a 3×3 matrix.

(b) Find $\mathbf{C}$.

[3]

27 Paper 1 – Systems of Equations

Example

Let $\mathbf{A}=\begin{pmatrix}1 & 2 & 3\\ 0 & -5 & -6\\ 0 & -3 & -2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}2\\ 7\\ 11\end{pmatrix}$ and $\mathbf{X}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$.

(a) Write down $\mathbf{A}^{-1}$.

[2]

(b) Solve $\mathbf{X}$ in the equation $\mathbf{AX}=\mathbf{B}$.

[3]

Solution

(a) $\mathbf{A}^{-1}=\begin{pmatrix}1 & \frac{5}{8} & -\frac{3}{8}\\ 0 & \frac{1}{4} & -\frac{3}{4}\\ 0 & -\frac{3}{8} & \frac{5}{8}\end{pmatrix}$ A2

[2]

(b) $\mathbf{AX}=\mathbf{B}$

$\mathbf{X}=\mathbf{A}^{-1}\mathbf{B}$ (M1) for valid approach

$\mathbf{X}=\begin{pmatrix}\frac{9}{4}\\ -\frac{13}{2}\\ \frac{17}{4}\end{pmatrix}$ A2

[3]

Exercise 27

1. Let $\mathbf{A} = \begin{pmatrix} 2 & 1 & 6 \\ 4 & 5 & 1 \\ 1 & 2 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

(a) Write down $\mathbf{A}^{-1}$.

[2]

(b) Solve $\mathbf{X}$ in the equation $\mathbf{AX} = \mathbf{B}$.

[3]

2. Let $\mathbf{A} = \begin{pmatrix} 2 & 0 & -5 \\ 1 & 2 & -8 \\ 6 & -1 & -3 \end{pmatrix}$.

(a) Write down $\mathbf{A}^{-1}$.

[2]

(b) Hence, solve the system $\begin{cases} 2x - 5z = 740 \\ x + 2y - 8z = 592 \\ 6x - y - 3z = -444 \end{cases}$.

[4]

3. Let $\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & a & 1 \\ 1 & 1 & 0 \end{pmatrix}$, where $a \in \mathbb{Z}$. It is given that $\mathbf{A}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix}$.

(a) Find a.

[2]

It is also given that $\mathbf{B} = \begin{pmatrix} 0 \\ 6 \\ 9 \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

(b) Solve $\mathbf{X}$ in the equation $\mathbf{AX} = \mathbf{B}$.

[3]

4. Let $\mathbf{A}=\begin{pmatrix} p & 16 & 16 \\ 8 & -8 & q \\ 8 & -16 & -16 \end{pmatrix}$, where p, $q\in\mathbb{Z}$. It is given that $\mathbf{A}^{-1}=\begin{pmatrix} \frac{1}{16} & 0 & \frac{1}{16} \\ \frac{1}{64} & \frac{1}{16} & -\frac{5}{64} \\ \frac{1}{64} & -\frac{1}{16} & \frac{3}{64} \end{pmatrix}$.

(a) Find p and q.

[3]

It is also given that $\mathbf{B}=\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{X}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

(b) Solve $\mathbf{X}$ in the equation $\mathbf{AX}=\mathbf{B}$.

[3]

28 Paper 1 – Transformation Matrices

Example

The transformation $\mathbf{S}$ and $\mathbf{T}$ are represented by the matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ respectively.

(a) Find $\mathbf{ST}$.

[2]

(b) Describe the transformation represented by $\mathbf{ST}$.

[1]

(c) $\mathbf{ST}$ transforms the point $(2, 4)$ to the point P. Find the coordinates of P.

[2]

Solution

(a) $\mathbf{ST} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

$\mathbf{ST} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ A2

[2]

(b) Rotation anticlockwise of $\frac{2\pi}{3}$ radians about the origin. A1

[1]

(c) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ (M1) for valid approach

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4.464101615 \\ -0.2679491924 \end{pmatrix}$

Thus, the coordinates of P are $(-4.46, -0.268)$. A1

[2]

7

Exercise 28

1. The transformation $\mathbf{S}$ and $\mathbf{T}$ are represented by the matrices $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$ respectively.

(a) Find $\mathbf{ST}$.

[2]

(b) Describe the transformation represented by $\mathbf{ST}$.

[1]

(c) $\mathbf{ST}$ transforms the point $(3, -5)$ to the point P. Find the coordinates of P.

[2]

2. The transformation $\mathbf{S}$ and $\mathbf{T}$ are represented by the matrices $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ respectively.

(a) Find $\mathbf{ST}$.

[2]

(b) Describe the transformation represented by $\mathbf{ST}$.

[1]

(c) $\mathbf{ST}$ transforms the point P to the point $(2, 1)$. Find the coordinates of P.

[3]

3. Let $\mathbf{T} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$.

(a) Describe the transformation represented by $\mathbf{T}$.

[1]

(b) $\mathbf{T}$ transforms the point $(4, 4)$ to the point P. Find the coordinates of P.

[2]

(c) Write down the smallest positive integer n such that $\mathbf{T}^n = \mathbf{I}$, where $\mathbf{I}$ is a 2×2 identity matrix.

[2]

4. Let $\mathbf{T} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$.

(a) Describe the transformation represented by $\mathbf{T}$.

[1]

(b) $\mathbf{T}$ transforms the point P to the point $(-2, 2\sqrt{3})$. Find the coordinates of P.

[3]

(c) Write down the smallest positive integer n such that $\mathbf{T}^n = \mathbf{I}$, where $\mathbf{I}$ is a 2×2 identity matrix.

[2]

7

29 Paper 2 – Eigenvalues and Eigenvectors

Example

The matrix $\mathbf{A}$ is defined by $\mathbf{A}=\begin{pmatrix}-2 & 1\\ -5 & 4\end{pmatrix}$. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{A}$, where $\lambda_1<\lambda_2$.

(a) Find the characteristic polynomial of $\mathbf{A}$.

[2]

(b) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{A}$ corresponding to λ_1 and λ_2 respectively.

(c) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

It is given that $\det(\mathbf{A})=\alpha\lambda_1\lambda_2$, where $\alpha\in\mathbb{R}$.

(d) Find α.

[2]

It is given that $\mathbf{A}^n=\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{P}$ is a 2×2 matrix and $\mathbf{D}$ is a 2×2 diagonal matrix.

(e) Write down

(i) $\mathbf{P}$;

(ii) $\mathbf{D}^n$.

[3]

(f) Hence, express $\mathbf{A}^n$ in terms of n.

[3]

Solution

(a) The characteristic polynomial of $\mathbf{A}$

$=\det(\mathbf{A}-\lambda\mathbf{I})$

$=\begin{vmatrix}-2-\lambda & 1\\ -5 & 4-\lambda\end{vmatrix}$ (M1) for valid approach

$=(-2-\lambda)(4-\lambda)-(1)(-5)$

$=-8+2\lambda-4\lambda+\lambda^2+5$

$=\lambda^2-2\lambda-3$ A1

[2]

(b) $\lambda_1=-1,\ \lambda_2=3$ A2

[2]

(c) $\mathbf{v}_1=\begin{pmatrix}1\\1\end{pmatrix},\ \mathbf{v}_2=\begin{pmatrix}1\\5\end{pmatrix}$ A2

[2]

(d) $\det(\mathbf{A})=\alpha\lambda_1\lambda_2$

$\therefore -3=\alpha(-1)(3)$ (M1) for setting equation

$\alpha=1$ A1

[2]

(e) (i) $\begin{pmatrix}1 & 1\\ 1 & 5\end{pmatrix}$ A1

(ii) $\begin{pmatrix}(-1)^n & 0\\ 0 & 3^n\end{pmatrix}$ A2

[3]

(f) $\mathbf{A}^n=\begin{pmatrix}1 & 1\\ 1 & 5\end{pmatrix}\begin{pmatrix}(-1)^n & 0\\ 0 & 3^n\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 5\end{pmatrix}^{-1}$

$\mathbf{A}^n=\begin{pmatrix}(-1)^n & 3^n\\ (-1)^n & 5\cdot 3^n\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 5\end{pmatrix}^{-1}$ A1

$\mathbf{A}^n=\begin{pmatrix}(-1)^n & 3^n\\ (-1)^n & 5\cdot 3^n\end{pmatrix}\begin{pmatrix}\frac{5}{4} & -\frac{1}{4}\\ -\frac{1}{4} & \frac{1}{4}\end{pmatrix}$ (A1) for correct approach

$\mathbf{A}^n=\begin{pmatrix}\frac{5}{4}(-1)^n-\frac{1}{4}\cdot 3^n & -\frac{1}{4}(-1)^n+\frac{1}{4}\cdot 3^n\\ \frac{5}{4}(-1)^n-\frac{5}{4}\cdot 3^n & -\frac{1}{4}(-1)^n+\frac{5}{4}\cdot 3^n\end{pmatrix}$ A1

[3]

Exercise 29

1. The matrix $\mathbf{A}$ is defined by $\mathbf{A}=\begin{pmatrix}-2 & -3\\ 1 & 2\end{pmatrix}$. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{A}$, where $\lambda_1<\lambda_2$.

(a) Find the characteristic polynomial of $\mathbf{A}$.

[2]

(b) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{A}$ corresponding to λ_1 and λ_2 respectively.

(c) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

It is given that $\det(\mathbf{A})=\dfrac{\alpha}{\lambda_1\lambda_2}$, where $\alpha\in\mathbb{R}$.

(d) Find α.

[2]

It is given that $\mathbf{A}^n=\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{P}$ is a 2×2 matrix and $\mathbf{D}$ is a 2×2 diagonal matrix.

(e) Write down

(i) $\mathbf{P}$;

(ii) $\mathbf{D}^n$.

[3]

(f) Hence, express $\mathbf{A}^n$ in terms of n.

[3]

2. The matrix $\mathbf{A}$ is defined by $\mathbf{A}=\begin{pmatrix}9 & -4\\ 2 & 3\end{pmatrix}$. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{A}$, where $\lambda_1<\lambda_2$.

(a) Find $\det(\mathbf{A}-\lambda\mathbf{I})$, giving the answer in terms of λ.

[2]

(b) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{A}$ corresponding to λ_1 and λ_2 respectively.

(c) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

It is given that $3\det(\mathbf{A})+\alpha\lambda_1\lambda_2=0$, where $\alpha\in\mathbb{R}$.

(d) Find α.

[2]

It is given that $\mathbf{A}^n=\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{P}$ is a 2×2 matrix and $\mathbf{D}$ is a 2×2 diagonal matrix.

(e) Write down

(i) $\mathbf{P}$;

(ii) $\mathbf{D}^n$.

[3]

(f) Hence, find $\mathbf{A}^{10}$, giving the entries in exact values.

[3]

3. The matrix $\mathbf{M}$ is defined by $\mathbf{M}=\begin{pmatrix}-1 & \frac{1}{16}\\ -35 & 2\end{pmatrix}$. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{M}$, where $\lambda_1<\lambda_2$.

(a) Find the characteristic polynomial of $\mathbf{M}$.

[2]

(b) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$ corresponding to λ_1 and λ_2 respectively.

(c) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

It is given that $\mathbf{M}^n=\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{P}$ is a 2×2 matrix and $\mathbf{D}$ is a 2×2 diagonal matrix.

(d) Write down

(i) $\mathbf{P}$;

(ii) $\mathbf{D}^n$.

[3]

(e) Hence, express $\mathbf{M}^n$ in terms of n.

[3]

Let $f(n)$ be the first diagonal entry of $\mathbf{M}^n$.

(f) Write down $\lim_{n\to\infty} f(n)$.

[1]

4. The matrix $\mathbf{M}$ is defined by $\mathbf{M}=\begin{pmatrix}\frac{3}{2} & -\frac{1}{2}\\ 1 & 0\end{pmatrix}$. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{M}$, where $\lambda_1<\lambda_2$.

(a) Find the characteristic polynomial of $\mathbf{M}$.

[2]

(b) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$ corresponding to λ_1 and λ_2 respectively.

(c) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

It is given that $\mathbf{M}^n=\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{P}$ is a 2×2 matrix and $\mathbf{D}$ is a 2×2 diagonal matrix.

(d) Write down

(i) $\mathbf{P}$;

(ii) $\mathbf{D}^n$.

[3]

(e) Hence, express $\mathbf{M}^n$ in terms of n.

[3]

Let $g(n)$ be the last diagonal entry of $\mathbf{M}^n$.

(f) Write down $\lim_{n\to\infty} g(n)$.

[1]

30 Paper 2 – Miscellaneous Problems

Example

The function f is defined by $f(x) = ax^2 + bx + c$, where a, b, $c \in \mathbb{Z}$. It is given that the graph of f passes through $(-10, 540)$, $(10, 500)$ and $(20, 1980)$.

(a) (i) Show that $100a - 10b + c = 540$.

(ii) Write down the other two equations in a, b and c.

[3]

The above three equations can be expressed in a matrix equation $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A}$ is a 3×3 matrix, and $\mathbf{X} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\mathbf{B}$ are two 3×1 matrices.

(b) Write down

(i) $\mathbf{A}$;

(ii) $\mathbf{B}$;

(iii) $\mathbf{A}^{-1}$.

[4]

(c) Hence, find the values of a, b and c.

[2]

(d) Find

(i) the equation of the axis of symmetry;

(ii) the y-coordinate of the vertex.

[4]

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Solution

(a)	(i)	$540 = a(-10)^2 + b(-10) + c$	A1
		$100a - 10b + c = 540$	AG
	(ii)	$100a + 10b + c = 500$	A1
		$400a + 20b + c = 1980$	A1

[3]

(b)	(i)	$\mathbf{A} = \begin{pmatrix} 100 & -10 & 1 \\ 100 & 10 & 1 \\ 400 & 20 & 1 \end{pmatrix}$	A1
	(ii)	$\mathbf{B} = \begin{pmatrix} 540 \\ 500 \\ 1980 \end{pmatrix}$	A1
	(iii)	$\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{600} & -\frac{1}{200} & \frac{1}{300} \\ -\frac{1}{20} & \frac{1}{20} & 0 \\ \frac{1}{3} & 1 & -\frac{1}{3} \end{pmatrix}$	A2

[4]

(c)	$a = 5$, $b = -2$ and $c = 20$	
	For any one correct answer	A1
	For all correct answers	A1

[2]

(d)	(i)	The equation of the axis of symmetry:	
		$x = -\frac{-2}{2(5)}$	(A1) for substitution
		$x = \frac{1}{5}$	A1
	(ii)	The y-coordinate of the vertex	
		$= 5\left(\frac{1}{5}\right)^2 - 2\left(\frac{1}{5}\right) + 20$	(M1) for substitution
		$= \frac{99}{5}$	A1

[4]

Exercise 30

1. The function f is defined by $f(x) = ax^2 + bx + c$, where a, b, $c \in \mathbb{Z}$. It is given that the graph of f passes through $(50, 3600)$, $(20, -900)$ and $(5, -1125)$.

(a) (i) Show that $2500a + 50b + c = 3600$.

(ii) Write down the other two equations in a, b and c.

[3]

The above three equations can be expressed in a matrix equation $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A}$ is a 3×3 matrix, and $\mathbf{X} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\mathbf{B}$ are two 3×1 matrices.

(b) Write down

(i) $\mathbf{A}$;

(ii) $\mathbf{B}$;

(iii) $\mathbf{A}^{-1}$.

[4]

(c) Hence, find the values of a, b and c.

[2]

(d) Find

(i) the x-intercept(s) of the graph of f;

(ii) the y-coordinate of the vertex.

[5]

2. The function f is defined by $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, $d \in \mathbb{Z}$. It is given that the graph of f passes through $(0, -384)$, $(2, -840)$, $(6, -2520)$ and $(10, -5544)$.

(a) (i) Show that $d = -384$.

(ii) Show that $4a + 2b + c = -228$.

(iii) Write down the other two equations in a, b and c.

[4]

The above three equations can be expressed in a matrix equation $\mathbf{AX}=\mathbf{B}$, where $\mathbf{A}$ is a 3×3 matrix, and $\mathbf{X}=\begin{pmatrix}a\\b\\c\end{pmatrix}$ and $\mathbf{B}$ are two 3×1 matrices.

(b) Write down

(i) $\mathbf{A}$;

(ii) $\mathbf{B}$;

(iii) $\mathbf{A}^{-1}$.

[4]

(c) Hence, find the values of a, b and c.

[2]

(d) Write down

(i) the x-intercept(s) of the graph of f;

(ii) the y-intercept of the graph of f.

[4]

3. Let $\mathbf{M}=\begin{pmatrix}1 & 0\\5 & 1\end{pmatrix}$.

(a) (i) Find $\mathbf{M}^2$.

(ii) Find $\mathbf{M}^3$.

(iii) By using the above results, write down $\mathbf{M}^{50}$.

[5]

Let $s(n)=\mathbf{M}+\mathbf{M}^2+\mathbf{M}^3+\cdots+\mathbf{M}^n$, where $n\geq1$.

(b) (i) Write down $s(2)$.

(ii) Write down $s(3)$.

(iii) By using (b)(i) and (b)(iii), find $s(50)$.

[6]

Let $p(n)=\mathbf{M}\times\mathbf{M}^2\times\mathbf{M}^3\times\cdots\times\mathbf{M}^n$, where $n\geq1$.

(c) Find $p(50)$.

[4]

4. Let $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$.

(a) (i) Find $\mathbf{A}^2$.

(ii) Find $\mathbf{A}^3$.

(iii) By using the above results, write down $\mathbf{A}^{30}$.

[5]

Let $\mathbf{B} = \mathbf{A} + \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}$.

(b) (i) Show that $\mathbf{B}^2 = \begin{pmatrix} 1 & 21 \\ 0 & 4 \end{pmatrix}$.

(ii) Find $\mathbf{B}^3$.

It is given that $\mathbf{B}^4$ can be expressed as $\begin{pmatrix} 1 & 7+14+28+56 \\ 0 & 16 \end{pmatrix}$.

(iii) Find $\mathbf{B}^{30}$.

[8]

(c) Explain why $\det(\mathbf{B}^n) = \det(\mathbf{A}^n) + \det\left(\begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}^n\right)$ is not always true for $n \geq 1$, $n \in \mathbb{Z}$.

[1]

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