Chapter 4 Solution

Exercise 14

1. (a) $\sqrt[3]{1+2x} = (1+2x)^{\frac{1}{3}}$

$$\sqrt[3]{1+2x} = 1 + \left(\frac{1}{3}\right)(2x) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}(2x)^2$$

$$+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}(2x)^3 + \cdots$$ M1A1

$$\sqrt[3]{1+2x} = 1 + \frac{2}{3}x - \frac{4}{9}x^2 + \frac{40}{81}x^3 + \cdots$$ A1

[3]

(b) $\sqrt[3]{1.06} = \sqrt[3]{1+2(0.03)}$

$$\sqrt[3]{1.06} \approx 1 + \frac{2}{3}(0.03) - \frac{4}{9}(0.03)^2 + \frac{40}{81}(0.03)^3$$ M1

$$\sqrt[3]{1.06} \approx 1.019613333$$

$$\sqrt[3]{1.06} \approx 1.02$$ A1

[2]

2. (a)

$$(1+bx)^{-\frac{2}{3}} = 1 + \left(-\frac{2}{3}\right)(bx) + \frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{2!}(bx)^2$$ M1A1

$$+\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{3!}(bx)^3 + \cdots$$

$$(1+bx)^{-\frac{2}{3}} = 1 - \frac{2}{3}bx + \frac{5}{9}b^2x^2 - \frac{40}{81}b^3x^3 + \cdots$$

$$\therefore -\frac{40}{81}b^3 = -\frac{2560}{81}$$ (A1) for correct equation

$$b^3 = 64$$

$$b = 4$$ A1

[4]

(b) The coefficient of x^2

$$= \frac{5}{9}(4)^2$$ A1

$$= \frac{80}{9}$$ A1

[2]

3. (a)

$$\sqrt{4-x}=\sqrt{4\left(1-\frac{1}{4}x\right)}$$

$$\sqrt{4-x}=2\left(1+\left(-\frac{1}{4}x\right)\right)^{\frac{1}{2}}$$

$$\sqrt{4-x}=2\left(\begin{array}{l}1+\left(\frac{1}{2}\right)\left(-\frac{1}{4}x\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(-\frac{1}{4}x\right)^2\\+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(-\frac{1}{4}x\right)^3+\cdots\end{array}\right)$$ M1A1

$$\sqrt{4-x}=2\left(1-\frac{1}{8}x-\frac{1}{128}x^2-\frac{1}{1024}x^3+\cdots\right)$$

$$\sqrt{4-x}=2-\frac{1}{4}x-\frac{1}{64}x^2-\frac{1}{512}x^3+\cdots$$ A1

[3]

(b)

$$\sqrt{\frac{15}{4}}=\sqrt{4-\frac{1}{4}}$$

$$\sqrt{\frac{15}{4}}\approx 2-\frac{1}{4}\left(\frac{1}{4}\right)-\frac{1}{64}\left(\frac{1}{4}\right)^2-\frac{1}{512}\left(\frac{1}{4}\right)^3$$ M1A1

$$\sqrt{\frac{15}{4}}\approx 1.93649292$$

$$\therefore \sqrt{15}\approx(1.93649292)(\sqrt{4})$$ A1

$$\sqrt{15}\approx 3.87298584$$

$$\sqrt{15}\approx 3.87$$ A1

[4]

4. (a) $\sqrt[4]{81+ax} = \sqrt[4]{81\left(1+\frac{a}{81}x\right)}$

$\sqrt[4]{81+ax} = 3\left(1+\left(\frac{a}{81}x\right)\right)^{\frac{1}{4}}$

$\sqrt[4]{81+ax} = 3\left(1+\left(\frac{1}{4}\right)\left(\frac{a}{81}x\right)+\frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)}{2!}\left(\frac{a}{81}x\right)^2 +\cdots\right)$ M1A1

$\sqrt[4]{81+ax} = 3\left(1+\frac{a}{324}x-\frac{a^2}{69984}x^2+\cdots\right)$

$\sqrt[4]{81+ax} = 3+\frac{a}{108}x-\frac{a^2}{23328}x^2+\cdots$ A1

[3]

(b) $\left(\frac{a}{108}\right)\left(-\frac{a^2}{23328}\right) = -0.5$ (A1) for correct equation

$-\frac{a^3}{2519424} = -0.5$

$a^3 = 1259712$

$a = 108$ A1

[2]

(c) $-1 < \frac{108}{81}x < 1$

$-\frac{3}{4} < x < \frac{3}{4}$ A2

[2]

Exercise 15

1. (a)

$$\frac{1}{(1+3x)^3}=(1+3x)^{-3}$$

$$\frac{1}{(1+3x)^3}=1+(-3)(3x)+\frac{(-3)(-4)}{2!}(3x)^2$$

$$+\frac{(-3)(-4)(-5)}{3!}(3x)^3+\cdots$$ M1A1

$$\frac{1}{(1+3x)^3}=1-9x+54x^2-270x^3+\cdots$$ A1

[3]

(b)

$$0.97^{-3}=\frac{1}{(1+3(-0.01))^3}$$

$$0.97^{-3}\approx 1-9(-0.01)+54(-0.01)^2-270(-0.01)^3$$ M1

$$0.97^{-3}\approx 1.09567$$

$$0.97^{-3}\approx 1.10$$ A1

[2]

2. (a)

$$(1+bx)^{-2}=1+(-2)(bx)+\frac{(-2)(-3)}{2!}(bx)^2$$

$$+\frac{(-2)(-3)(-4)}{3!}(bx)^3+\cdots$$ M1A1

$$(1+bx)^{-2}=1-2bx+3b^2x^2-4b^3x^3+\cdots$$

$$\therefore -4b^3=2048$$ (A1) for correct equation

$$b^3=-512$$

$$b=-8$$ A1

[4]

(b) The coefficient of x^2

$$=3(-8)^2$$ A1

$$=192$$ A1

[2]

3. (a)

$$\frac{1}{4+12x+9x^2}=\frac{1}{(2+3x)^2}$$

$$\frac{1}{4+12x+9x^2}=\frac{1}{2^2\left(1+\frac{3}{2}x\right)^2}$$

$$\frac{1}{4+12x+9x^2}=\frac{1}{4}\left(1+\frac{3}{2}x\right)^{-2}$$

$$\frac{1}{4+12x+9x^2}=\frac{1}{4}\left(\begin{array}{l}1+(-2)\left(\frac{3}{2}x\right)+\frac{(-2)(-3)}{2!}\left(\frac{3}{2}x\right)^2\\+\frac{(-2)(-3)(-4)}{3!}\left(\frac{3}{2}x\right)^3+\cdots\end{array}\right)$$ M1A1

$$\frac{1}{4+12x+9x^2}=\frac{1}{4}\left(1-3x+\frac{27}{4}x^2-\frac{27}{2}x^3+\cdots\right)$$

$$\frac{1}{4+12x+9x^2}=\frac{1}{4}-\frac{3}{4}x+\frac{27}{16}x^2-\frac{27}{8}x^3+\cdots$$ A1

[3]

(b)

$$\frac{1}{2.15^2}=\frac{1}{(2+3(0.05))^2}$$

$$\frac{1}{2.15^2}\approx\frac{1}{4}-\frac{3}{4}(0.05)+\frac{27}{16}(0.05)^2-\frac{27}{8}(0.05)^3$$ M1A1

$$\frac{1}{2.15^2}\approx 0.216296875$$

$$\therefore -\frac{7}{2.15^2}\approx(0.216296875)(-7)$$ A1

$$-\frac{7}{2.15^2}\approx -1.514078125$$

$$-\frac{7}{2.15^2}\approx -1.51$$ A1

[4]

4. (a) $\dfrac{2-x}{a+x}=(2-x)(a+x)^{-1}$

$\dfrac{2-x}{a+x}=(2-x)(a)^{-1}\left(1+\dfrac{1}{a}x\right)^{-1}$

$\dfrac{2-x}{a+x}=\dfrac{2-x}{a}\left(1+(-1)\left(\dfrac{1}{a}x\right)+\dfrac{(-1)(-2)}{2!}\left(\dfrac{1}{a}x\right)^2+\cdots\right)$ M1A1

$\dfrac{2-x}{a+x}=\left(\dfrac{2}{a}-\dfrac{x}{a}\right)\left(1-\dfrac{1}{a}x+\dfrac{1}{a^2}x^2+\cdots\right)$

$\dfrac{2-x}{a+x}=\dfrac{2}{a}-\dfrac{2}{a^2}x+\dfrac{2}{a^3}x^2-\dfrac{1}{a}x+\dfrac{1}{a^2}x^2-\dfrac{1}{a^3}x^3+\cdots$

$\dfrac{2-x}{a+x}=\dfrac{2}{a}+\left(-\dfrac{1}{a}-\dfrac{2}{a^2}\right)x+\left(\dfrac{1}{a^2}+\dfrac{2}{a^3}\right)x^2+\cdots$ A1

[3]

(b) (i) $\dfrac{2}{a}+\left(-\dfrac{1}{a}-\dfrac{2}{a^2}\right)=\dfrac{1}{9}$ (A1) for correct equation

$\dfrac{1}{a}-\dfrac{2}{a^2}=\dfrac{1}{9}$

$\dfrac{a-2}{a^2}=\dfrac{1}{9}$

$9a-18=a^2$

$a^2-9a+18=0$

$(a-3)(a-6)=0$

$a=3$ or $a=6$ (*Rejected*) A1

(ii) $\dfrac{5}{27}$ A1

[3]

(c) $-1<\dfrac{1}{3}x<1$

$-3<x<3$ A2

[2]

Exercise 16

1. (a) Let $\frac{2}{(2+x)(5+x)} \equiv \frac{A}{2+x} + \frac{B}{5+x}$, where A and B are constants.

$\frac{2}{(2+x)(5+x)} \equiv \frac{A(5+x)}{(2+x)(5+x)} + \frac{B(2+x)}{(2+x)(5+x)}$ M1

$\frac{2}{(2+x)(5+x)} \equiv \frac{5A+Ax+2B+Bx}{(2+x)(5+x)}$

$2 \equiv (A+B)x + (5A+2B)$ A1

$0 = A+B$

$-B = A$

$2 = 5A+2B$

$\therefore 2 = 5(-B)+2B$ A1

$2 = -5B+2B$

$2 = -3B$

$B = -\frac{2}{3}$

$\therefore A = \frac{2}{3}$

$\therefore \frac{2}{(2+x)(5+x)} \equiv \frac{2}{3(2+x)} - \frac{2}{3(5+x)}$ A1

[4]

(b) $$\frac{2}{(2+x)(5+x)}=\frac{2}{3(2+x)}-\frac{2}{3(5+x)}$$

$$\frac{2}{(2+x)(5+x)}=\frac{2}{6}\left(1+\frac{1}{2}x\right)^{-1}-\frac{2}{15}\left(1+\frac{1}{5}x\right)^{-1}$$

$$\frac{2}{(2+x)(5+x)}$$

$$=\frac{1}{3}\left(1+(-1)\left(\frac{1}{2}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{1}{2}x\right)^2+\cdots\right)$$ M1A1

$$-\frac{2}{15}\left(1+(-1)\left(\frac{1}{5}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{1}{5}x\right)^2+\cdots\right)$$

$$\frac{2}{(2+x)(5+x)}=\frac{1}{3}\left(1-\frac{1}{2}x+\frac{1}{4}x^2+\cdots\right)$$

$$-\frac{2}{15}\left(1-\frac{1}{5}x+\frac{1}{25}x^2+\cdots\right)$$

$$\frac{2}{(2+x)(5+x)}$$ A1

$$=\frac{1}{3}-\frac{1}{6}x+\frac{1}{12}x^2-\frac{2}{15}+\frac{2}{75}x-\frac{2}{375}x^2+\cdots$$

$$\frac{2}{(2+x)(5+x)}=\frac{1}{5}-\frac{7}{50}x+\frac{39}{500}x^2+\cdots$$

The required sum

$$=-\frac{7}{50}+\frac{39}{500}$$

$$=-\frac{31}{500}$$ A1

[4]

2. (a) Let $\dfrac{1+x}{(4+x)^2} \equiv \dfrac{A}{4+x} + \dfrac{B}{(4+x)^2}$, where A and B are constants.

$$\frac{1+x}{(4+x)^2} \equiv \frac{A(4+x)}{(4+x)^2} + \frac{B}{(4+x)^2}$$ M1

$$\frac{1+x}{(4+x)^2} \equiv \frac{4A+Ax+B}{(4+x)^2}$$

$$1+x \equiv Ax+(4A+B)$$ A1

$$A=1$$
$$1=4A+B$$
$$\therefore 1=4(1)+B$$ A1

$$1=4+B$$
$$B=-3$$

$$\therefore \frac{1+x}{(4+x)^2} \equiv \frac{1}{4+x} - \frac{3}{(4+x)^2}$$ A1

[4]

(b) $\dfrac{1+x}{(4+x)^2}=\dfrac{1}{4+x}-\dfrac{3}{(4+x)^2}$

$$\frac{1+x}{(4+x)^2}=\frac{1}{4}\left(1+\frac{1}{4}x\right)^{-1}-\frac{3}{16}\left(1+\frac{1}{4}x\right)^{-2}$$

$$\frac{1+x}{(4+x)^2}=\frac{1}{4}\left(\begin{array}{l}1+(-1)\left(\frac{1}{4}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{1}{4}x\right)^2\\ +\frac{(-1)(-2)(-3)}{3!}\left(\frac{1}{4}x\right)^3+\cdots\end{array}\right)$$

M1A1

$$-\frac{3}{16}\left(\begin{array}{l}1+(-2)\left(\frac{1}{4}x\right)+\frac{(-2)(-3)}{2!}\left(\frac{1}{4}x\right)^2\\ +\frac{(-2)(-3)(-4)}{3!}\left(\frac{1}{4}x\right)^3+\cdots\end{array}\right)$$

$$\frac{1+x}{(4+x)^2}=\frac{1}{4}\left(1-\frac{1}{4}x+\frac{1}{16}x^2-\frac{1}{64}x^3+\cdots\right)$$

M1

$$-\frac{3}{16}\left(1-\frac{1}{2}x+\frac{3}{16}x^2-\frac{1}{16}x^3+\cdots\right)$$

$$\frac{1+x}{(4+x)^2}=\frac{1}{4}-\frac{1}{16}x+\frac{1}{64}x^2-\frac{1}{256}x^3$$

A1

$$-\frac{3}{16}+\frac{3}{32}x-\frac{9}{256}x^2+\frac{3}{256}x^3+\cdots$$

$$\frac{1+x}{(4+x)^2}=\frac{1}{16}+\frac{1}{32}x-\frac{5}{256}x^2+\frac{1}{128}x^3+\cdots$$

Thus, the coefficient of x^3 is $\dfrac{1}{128}$. A1

[5]

3. (a) Let $\frac{9x}{(3+x)(6-x)} \equiv \frac{A}{3+x} + \frac{B}{6-x}$, where A and B are constants.

$\frac{9x}{(3+x)(6-x)} \equiv \frac{A(6-x)}{(3+x)(6-x)} + \frac{B(3+x)}{(3+x)(6-x)}$ M1

$\frac{9x}{(3+x)(6-x)} \equiv \frac{6A - Ax + 3B + Bx}{(3+x)(6-x)}$

$9x \equiv (-A+B)x + (6A+3B)$ A1

$9 = -A + B$

$B = 9 + A$

$0 = 6A + 3B$

$\therefore 0 = 6A + 3(9+A)$ A1

$0 = 9A + 27$

$-27 = 9A$

$A = -3 \therefore B = 6$

$\therefore \frac{9x}{(3+x)(6-x)} \equiv -\frac{3}{3+x} + \frac{6}{6-x}$ A1

$\frac{9x}{(3+x)(6-x)} = -\frac{3}{3}\left(1+\frac{1}{3}x\right)^{-1} + \frac{6}{6}\left(1-\frac{1}{6}x\right)^{-1}$

$\frac{9x}{(3+x)(6-x)}$

$= -\left(1 + (-1)\left(\frac{1}{3}x\right) + \frac{(-1)(-2)}{2!}\left(\frac{1}{3}x\right)^2 + \cdots\right)$ M1A1

$+\left(1 + (-1)\left(-\frac{1}{6}x\right) + \frac{(-1)(-2)}{2!}\left(-\frac{1}{6}x\right)^2 + \cdots\right)$

$\frac{9x}{(3+x)(6-x)} = -\left(1 - \frac{1}{3}x + \frac{1}{9}x^2 + \cdots\right)$

$+\left(1 + \frac{1}{6}x + \frac{1}{36}x^2 + \cdots\right)$

$\frac{9x}{(3+x)(6-x)} = -1 + \frac{1}{3}x - \frac{1}{9}x^2 + 1 + \frac{1}{6}x + \frac{1}{36}x^2 + \cdots$

$\frac{9x}{(3+x)(6-x)} = \frac{1}{2}x - \frac{1}{12}x^2 + \cdots$ A1

[7]

(b) $-1 < \frac{1}{3}x < 1$ and $-1 < -\frac{1}{6}x < 1$

$-3 < x < 3$ and $-6 < x < 6$

$\therefore -3 < x < 3$ A2

[2]

4. Let $\dfrac{5x+3-\dfrac{30}{k}}{(3+kx)^2} \equiv \dfrac{A}{3+kx} + \dfrac{B}{(3+kx)^2}$, where A and B are constants.

$$\frac{5x+3-\frac{30}{k}}{(3+kx)^2} \equiv \frac{A(3+kx)}{(3+kx)^2} + \frac{B}{(3+kx)^2}$$ M1

$$\frac{5x+3-\frac{30}{k}}{(3+kx)^2} \equiv \frac{3A+Akx+B}{(3+kx)^2}$$

$$5x+3-\frac{30}{k} \equiv Akx + (3A+B)$$ A1

$$5 = Ak$$

$$A = \frac{5}{k}$$

$$3-\frac{30}{k} = 3A+B$$

$$\therefore 3-\frac{30}{k} = 3\left(\frac{5}{k}\right)+B$$ A1

$$\frac{3k-30}{k} = \frac{15}{k}+B$$

$$B = \frac{3k-45}{k}$$

$$\therefore \frac{5x+3-\frac{30}{k}}{(3+kx)^2} \equiv \frac{5}{k(3+kx)} + \frac{3k-45}{k(3+kx)^2}$$

$$\frac{5x+3-\frac{30}{k}}{(3+kx)^2} \equiv \frac{5}{3k\left(1+\frac{k}{3}x\right)} + \frac{3k-45}{9k\left(1+\frac{k}{3}x\right)^2}$$

$$\frac{5x+3-\frac{30}{k}}{(3+kx)^2} = \frac{5}{3k}\left(1+\frac{k}{3}x\right)^{-1} + \frac{k-15}{3k}\left(1+\frac{k}{3}x\right)^{-2}$$

$$\frac{5x+3-\frac{30}{k}}{(3+kx)^2}$$

$$= \frac{5}{3k}\left(1+(-1)\left(\frac{k}{3}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{k}{3}x\right)^2+\cdots\right)$$ M1A1

$$+\frac{k-15}{3k}\left(1+(-2)\left(\frac{k}{3}x\right)+\frac{(-2)(-3)}{2!}\left(\frac{k}{3}x\right)^2+\cdots\right)$$

$$\frac{5x+3-\frac{30}{k}}{(3+kx)^2}=\frac{5}{3k}\left(1-\frac{k}{3}x+\frac{k^2}{9}x^2+\cdots\right)$$

$$+\frac{k-15}{3k}\left(1-\frac{2k}{3}x+\frac{k^2}{3}x^2+\cdots\right)$$

$$\therefore \frac{5}{3k}\left(\frac{k^2}{9}\right)+\frac{k-15}{3k}\left(\frac{k^2}{3}\right)=-\frac{37}{27}$$ A1

$$\frac{5k}{27}+\frac{k^2-15k}{9}=-\frac{37}{27}$$

$$5k+3k^2-45k=-37$$

$$3k^2-40k+37=0$$ A1

$$(3k-37)(k-1)=0$$

$$k=\frac{37}{3}\ (\textit{Rejected})\text{ or } k=1$$ A1

[8]