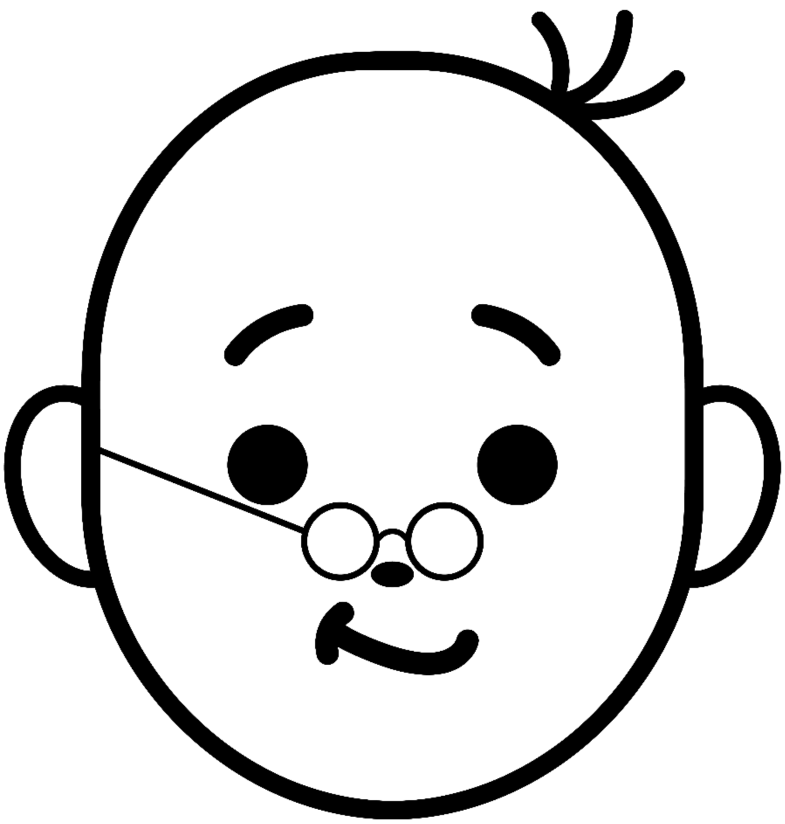
Your Intensive Notes Analysis and Approaches Standard Level for IBDP Mathematics



Algebra

Topics Covered

1 Standard Form

Important Notes

Standard form: A number in the format $\pm a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$ (k is an integer)

Notes on GDC		
TEXAS TI-84 Plus CE mode→SCI on the 2nd row to express any number in its standard form	**TEXAS TI-Nspire CX** Doc→Setting & Status →Document Settings… →Scientific on the Exponential Format row to express any number in its standard form	**CASIO fx-CG50** SHIFT MENU →Sci on the Display row →3 to express any number in its standard form

Example 1.1

A rectangle is 2376 cm long and 693 cm wide.

(a) Find the diagonal length of the rectangle, giving the answer in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$.

[2]

The required diagonal length

$= \sqrt{2376^2 + 693^2}$ Pythagoras' theorem (A1)

$= 2475\text{ cm}$

$= 2.475 \times 10^3\text{ cm}$ $a = 2.475$ & $k = 3$ (A1)

(b) Find the area of the rectangle, giving the answer correct to two significant figures and in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$.

[2]

The required area

$= (2376)(693)$ Base Length × Height (A1)

$= 1646568\text{ cm}^2$

$= 1600000\text{ cm}^2$ Round off to 2 sig. fig.

$= 1.6 \times 10^6\text{ cm}^2$ $a = 1.6$ & $k = 6$ (A1)

Exercise 1.1

The base length and the height of a right-angled triangle are 1107 mm and 4920 mm respectively.

(a) Find the length of the hypotenuse of the triangle, giving the answer in the form $a\times10^k$, where $1\le a<10$ and $k\in\mathbb{Z}$.

[2]

(b) Using (a), find the perimeter of the triangle, giving the answer correct to two significant figures and in the form $a\times10^k$, where $1\le a<10$ and $k\in\mathbb{Z}$.

[2]

(c) Find the area of the triangle, giving the answer correct to four significant figures and in the form $a\times10^k$, where $1\le a<10$ and $k\in\mathbb{Z}$.

[2]

2 Arithmetic Sequences

Important Notes

An arithmetic sequence is a sequence such that the next term is generated by adding or subtracting the same number from the previous term.

Properties of an arithmetic sequence $u_1, u_2, u_3, \cdots$:

1. u_1: First term
2. $d = u_2 - u_1 = u_n - u_{n-1}$: Common difference
3. $u_n = u_1 + (n-1)d$: General term (n th term)
4. $S_n = \frac{n}{2}[2u_1 + (n-1)d] = \frac{n}{2}[u_1 + u_n]$: Sum of the first n terms

$\sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + \cdots + u_{n-1} + u_n$: Summation sign

Notes on GDC		
TEXAS TI-84 Plus CE [y=] to input the general term →[2nd] [window] to set the starting row →[2nd] [graph] to find the value of the term needed	**TEXAS TI-Nspire CX** Graph to input the general term to generate a table →[ctrl] [T] to generate a table →[menu] [2] [5] to set the starting row to find the value of the term needed	**CASIO fx-CG50** Table to input the general term to generate a table →[F5] to set the starting row →[F6] to find the value of the term needed

Example 1.2

Consider the arithmetic sequence $15, 21, 27, \cdots$.

(a) Write down d, the common difference of this sequence.

[1]

$d = 21 - 15$	$d = u_2 - u_1$
$d = 6$	(A1)

(b) Find u_8.

[2]

$u_8 = u_1 + (8-1)d$	$u_n = u_1 + (n-1)d$
$u_8 = 15 + (8-1)(6)$	Correct approach (A1)
$u_8 = 15 + 42$	
$u_8 = 57$	(A1)

(c) Find n such that $u_n = 75$.

[2]

$u_n = 75$	Set up an equation
$\therefore 15 + (n-1)(6) = 75$	Correct equation (A1)
$6(n-1) = 60$	
$n - 1 = 10$	
$n = 11$	(A1)

(d) Find an expression of the sum of the first n terms of this sequence.

[3]

The sum of the first n terms

$= S_n$	
$= \frac{n}{2}\left[2u_1 + (n-1)d\right]$	$S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$ (M1)
$= \frac{n}{2}\left[2(15) + (n-1)(6)\right]$	$u_1 = 15$ & $d = 6$ (A1)
$= \frac{n}{2}(30 + 6n - 6)$	
$= \frac{n}{2}(6n + 24)$	
$= 3n^2 + 12n$	(A1)

(e) Hence, find the sum of the first ten terms of this sequence.

[2]

The required sum

$= S_{10}$

$= 3(10)^2 + 12(10)$ $n = 10$ (M1)

$= 300 + 120$

$= 420$ (A1)

Exercise 1.2

Consider the arithmetic sequence $\frac{39}{10}, \frac{38}{10}, \frac{37}{10}, \cdots$.

(a) Write down d, the common difference of this sequence.

[1]

(b) Find u_{12}.

[2]

(c) Find n such that $u_n = \frac{7}{10}$.

[2]

(d) Find an expression of the sum of the first n terms of this sequence.

[3]

(e) Hence, find the sum of the first ten terms of this sequence.

[2]

Example 1.3

Consider the arithmetic sequence with first term 70 and common difference -3.5. It is given that the r th term of the sequence is zero.

(a) Find r.

[2]

$u_r = 0$ — Set up an equation

$\therefore 70 + (r-1)(-3.5) = 0$ — Correct equation (A1)

$-3.5(r-1) = -70$

$r - 1 = 20$

$r = 21$ — (A1)

(b) Find the maximum value of the sum of the first n terms of this sequence.

[3]

The sum of the first n terms

$= S_n$

$= \frac{n}{2}\left[2u_1 + (n-1)d\right]$ — $S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$ (M1)

$= \frac{n}{2}\left[2(70) + (n-1)(-3.5)\right]$

$= \frac{n}{2}(140 - 3.5n + 3.5)$

$= \frac{n(143.5 - 3.5n)}{2}$

By considering the graph of $y = \frac{n(143.5 - 3.5n)}{2}$,

the coordinates of the maximum point are $(20.5, 735.4375)$, and the graph passes through $(20, 735)$ and $(21, 735)$. — GDC approach (M1)

Thus, the maximum value is 735. — (A1)

Exercise 1.3

Consider the arithmetic sequence with first term 120 and common difference -1.25. It is given that the mth term of the sequence is zero.

(a) Find m.

[2]

(b) Find the maximum value of the sum of the first n terms of this sequence.

[3]

3 Geometric Sequences

Important Notes

A geometric sequence is a sequence such that the next term is generated by multiplying or being divided by the same number from the previous term.

Properties of a geometric sequence $u_1, u_2, u_3, \cdots$:

1. u_1: First term
2. $r = u_2 \div u_1 = u_n \div u_{n-1}$: Common ratio
3. $u_n = u_1 \times r^{n-1}$: General term (nth term)
4. $S_n = \frac{u_1(1-r^n)}{1-r} = \frac{u_1(r^n-1)}{r-1}$: Sum of the first n terms
5. $S_\infty = u_1 + u_2 + u_3 + \cdots + u_n + \cdots = \frac{u_1}{1-r}$: Sum of infinite number of terms (Sum to infinity), valid only when $-1 < r < 1$

Properties of compound interest:

1. PV: Present value
2. $r\%$: Nominal annual interest rate
3. k: Number of compounded periods in one year
4. n: Number of years
5. $FV = PV\left(1+\frac{r\%}{k}\right)^{kn}$: Future value
6. $I = FV - PV$: Interest

Notes on GDC		
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Example 1.4

Consider the sequence $\ln x, a\ln x, \frac{1}{2}\ln x, \cdots$, $x>0$, $a\neq 0$, x, $a\in\mathbb{R}$.

(a) Consider the case where the sequence is arithmetic.

(i) Find a.

[2]

$a\ln x-\ln x=\frac{1}{2}\ln x-a\ln x$ $\quad$ $d=u_2-u_1=u_3-u_2$ (A1)

$a-1=\frac{1}{2}-a$

$2a=\frac{3}{2}$

$a=\frac{3}{4}$ $\quad$ (A1)

(ii) Hence, express the common difference in terms of $\ln x$.

[2]

The common difference

$=\frac{3}{4}\ln x-\ln x$ $\quad$ $d=u_2-u_1$ (A1)

$=-\frac{1}{4}\ln x$ $\quad$ (A1)

(b) Consider the case where the sequence is geometric.

(i) Find the possible values of a.

[3]

$a\ln x\div\ln x=\frac{1}{2}\ln x\div a\ln x$ $\quad$ $r=u_2\div u_1=u_3\div u_2$ (A1)

$a=\frac{1}{2a}$

$2a^2=1$

$a^2=\frac{1}{2}$

$a=\frac{1}{\sqrt{2}}$ or $a=-\frac{1}{\sqrt{2}}$ $\quad$ (A1)(A1)

(ii) Hence, find an expression of the sum of the first n terms of this sequence if $a>0$, giving the answer in terms of $\ln x$ and 2^M, $M \in \mathbb{R}$.

[4]

The sum of the first n terms

$= S_n$

$= \dfrac{u_1(1-r^n)}{1-r}$ — $S_n = \dfrac{u_1(1-r^n)}{1-r}$ (M1)

$= \dfrac{(\ln x)\left(1-\left(\frac{1}{\sqrt{2}}\right)^n\right)}{1-\frac{1}{\sqrt{2}}}$ — $u_1 = \ln x$ & $r = \dfrac{1}{\sqrt{2}}$ (A1)

$= \dfrac{(\ln x)\left(1-\left(\frac{1}{2^{0.5}}\right)^n\right)}{1-\frac{1}{2^{0.5}}}$ — $\sqrt{2} = 2^{0.5}$ (M1)

$= \dfrac{(\ln x)(1-(2^{-0.5})^n)}{1-2^{-0.5}}$

$= \dfrac{1-2^{-0.5n}}{1-2^{-0.5}}\ln x$ — (A1)

It is given that $S_\infty = 2-\sqrt{2}$ when $a<0$.

(iii) Find x.

[3]

$S_\infty = 2-\sqrt{2}$ — Set up an equation

$\therefore \dfrac{\ln x}{1-\left(-\frac{1}{\sqrt{2}}\right)} = 2-\sqrt{2}$ — Correct equation (A1)

$\ln x = (2-\sqrt{2})\left(1+\dfrac{1}{\sqrt{2}}\right)$

$\ln x = 2+\sqrt{2}-\sqrt{2}-1$

$\ln x = 1$ — Simplify the R.H.S. (A1)

$x = e^1$

$x = e$ — (A1)

Exercise 1.4

Consider the sequence $\log x, k\log x, \frac{1}{5}\log x, \cdots$, $x>0$, $k\neq 0$, x, $k\in\mathbb{R}$.

(a) Consider the case where the sequence is arithmetic.

(i) Find k.

[2]

(ii) Hence, express the common difference in terms of $\log x$.

[2]

(b) Consider the case where the sequence is geometric.

(i) Find the possible values of k.

[3]

(ii) Hence, find an expression of the sum of the first n terms of this sequence if $k < 0$, giving the answer in terms of $\log x$ and 5^M, $M \in \mathbb{R}$.

[4]

It is given that $S_\infty = \frac{5+\sqrt{5}}{2}$ when $k > 0$.

(iii) Find x.

[3]

Example 1.5

Consider the geometric sequence $\frac{1}{243}, \frac{1}{81}, \frac{1}{27}, \cdots$.

(a) Write down r, the common ratio of this sequence.

[1]

$r = \frac{1}{81} \div \frac{1}{243}$ — $r = u_2 \div u_1$

$r = 3$ — (A1)

(b) Find u_{12}.

[2]

$u_{12} = u_1 \times r^{12-1}$ — $u_n = u_1 \times r^{n-1}$

$u_{12} = \frac{1}{243} \times 3^{12-1}$ — $u_1 = \frac{1}{243}$ & $r = 3$ (A1)

$u_{12} = 729$ — (A1)

(c) Find n such that $u_n = 81$.

[2]

$u_n = 81$ — Set up an equation

$\therefore \frac{1}{243} \times 3^{n-1} = 81$ — Correct equation (A1)

$\frac{1}{243} \times 3^{n-1} - 81 = 0$

By considering the graph of $y = \frac{1}{243} \times 3^{n-1} - 81$,

the horizontal intercept is 10.

$\therefore n = 10$ — (A1)

(d) Find an expression of the sum of the first n terms of this sequence.

[3]

The sum of the first n terms

$= S_n$

$= \frac{u_1(1-r^n)}{1-r}$ — $S_n = \frac{u_1(1-r^n)}{1-r}$ (M1)

$= \frac{\frac{1}{243}(1-3^n)}{1-3}$ — $u_1 = \frac{1}{243}$ & $r = 3$ (A1)

$= -\frac{1}{486}(1-3^n)$ — (A1)

(e) Hence, find the sum of the first twelve terms of this sequence.

[2]

The required sum

$= S_{12}$

$= -\frac{1}{486}(1-3^{12})$ — $n = 12$ (M1)

$= 1093.497942$

$= 1090$ — (A1)

(f) Explain why the sum to infinity of this geometric sequence does not exist.

[1]

The common ratio is 3 which is not between -1 and 1. — (R1)

(g) Find the least value of n such that $S_n > 10000$.

[3]

$S_n > 10000$ — Set up an inequality

$\therefore -\frac{1}{486}(1-3^n) > 10000$ — Correct inequality (A1)

$-\frac{1}{486}(1-3^n) - 10000 > 0$

By considering the graph of $y = -\frac{1}{486}(1-3^n) - 10000$, the graph is above the horizontal axis when $n > 14.014543$. — GDC approach (M1)

$\therefore$ The least value of n is 15. — (A1)

Exercise 1.5

Consider the geometric sequence $\frac{7}{12}, \frac{7}{16}, \frac{21}{48}, \cdots$.

(a) Write down r, the common ratio of this sequence.

[1]

(b) Find u_7.

[2]

(c) Find n such that $u_n = \frac{189}{1024}$.

[2]

(d) Find an expression of the sum of the first n terms of this sequence.

[3]

(e) Hence, find the sum of the first fifteen terms of this sequence.

[2]

(f) Explain why the sum to infinity of this geometric sequence exists.

[1]

(g) Find the greatest value of n such that $S_n < 2.3315$.

[3]

Example 1.6

On 1st January 2024, Judy invests $\$P$ in an account that pays a nominal annual interest rate of 6%, compounded **half-yearly**. The amount of money in her account at the end of each year follows a geometric sequence with common ratio R.

(a) Find the exact value of R.

[3]

The amount of money after one year

$$= P\left(1+\frac{6\%}{2}\right)^{(2)(1)}$$ $FV = PV\left(1+\frac{r\%}{k}\right)^{kn}$ (M1)

$$\therefore R = \left(1+\frac{6\%}{2}\right)^{(2)(1)}$$ $R = \frac{FV}{PV}$ (A1)

$R = 1.0609$ (A1)

It is given that there is no further deposit to or any withdrawal from the account.

(b) Find the year in which the amount of money in Judy's account will become double the amount she invested.

[3]

$FV = 2P$ Set up an equation

$$\therefore P\left(1+\frac{6\%}{2}\right)^{2n} = 2P$$ Correct equation (A1)

$$\left(1+\frac{6\%}{2}\right)^{2n} = 2$$

$$\left(1+\frac{6\%}{2}\right)^{2n} - 2 = 0$$

By considering the graph of

$y = \left(1+\frac{6\%}{2}\right)^{2n} - 2$, the horizontal intercept is

11.724886. GDC approach (M1)

$\therefore$ The required year is 2035. (A1)

It is given that $P = 22000$.

(c) (i) Find the amount that Judy will have in her account after 3 years.

[3]

The amount

$= PV\left(1+\frac{r\%}{k}\right)^{kn}$ $\quad FV = PV\left(1+\frac{r\%}{k}\right)^{kn}$ (M1)

$= 22000\left(1+\frac{6\%}{2}\right)^{(2)(3)}$ $\quad r = 6$, $k = 2$ & $n = 3$ (A1)

$= \$26269.15052$

$= \$26300$ (A1)

(ii) Hence, find the interest that Judy can earn after 3 years.

[2]

The interest

$= 26269.15052 - 22000$ $\quad I = FV - PV$ (M1)

$= \$4269.150524$

$= \$4270$ (A1)

Starting from 1st January 2024, Judy's friend Cady invests $\$16000$ in an account that pays a nominal annual interest rate of 8%, compounded **monthly**. It is given that the amount of money in Cady's account will exceed that in Judy's account after several years.

(d) Find the minimum number of complete years when the amount of money in Cady's account exceeds that in Judy's account.

[3]

Let t be the number of years required for the amount of money in Cady's account exceeding that in Judy's account.

$16000\left(1+\frac{8\%}{12}\right)^{12t} > 22000\left(1+\frac{6\%}{2}\right)^{2t}$ $\quad$ Correct inequality (A1)

$16000\left(1+\frac{8\%}{12}\right)^{12t} - 22000\left(1+\frac{6\%}{2}\right)^{2t} > 0$

By considering the graph of

$y = 16000\left(1+\frac{8\%}{12}\right)^{12t} - 22000\left(1+\frac{6\%}{2}\right)^{2t}$, the graph is above the horizontal axis when $t > 15.446241$. $\quad$ GDC approach (M1)

$\therefore$ The minimum number of complete years is 16. (A1)

(e) Find the year when the sum of the amount of money in Cady's account and that in Judy's account first reaches $\$70000$.

[3]

Let T be the number of years required for the sum of the amount of money in Cady's account and that in Judy's account first reaches $\$70000$.

$$16000\left(1+\frac{8\%}{12}\right)^{12T}+22000\left(1+\frac{6\%}{2}\right)^{2T}\geq 70000$$ Correct inequality (A1)

$$16000\left(1+\frac{8\%}{12}\right)^{12T}+22000\left(1+\frac{6\%}{2}\right)^{2T}-70000\geq 0$$

By considering the graph of

$$y=16000\left(1+\frac{8\%}{12}\right)^{12T}+22000\left(1+\frac{6\%}{2}\right)^{2T}-70000$$

, the graph is above the horizontal axis when $T>8.9489625$. GDC approach (M1)

$\therefore$ The required year is 2032. (A1)

Exercise 1.6

On 1st January 2025, April invests $\$P$ in an account that pays a nominal annual interest rate of 8%, compounded **quarterly**. The amount of money in her account at the end of each year follows a geometric sequence with common ratio R.

(a) Find the exact value of R.

[3]

It is given that there is no further deposit to or any withdrawal from the account.

(b) Find the year in which the amount of money in April's account will become 2.5 times the amount she invested.

[3]

It is given that $P = 10000$.

(c) (i) Find the amount that April will have in her account after 4 years.

[3]

(ii) Hence, find the interest that April can earn after 4 years.

[2]

Starting from 1st January 2025, April's sister Bea invests $\$15000$ in an account that pays a nominal annual interest rate of 3%, compounded **half-yearly**. It is given that the amount of money in April's account will exceed that in Bea's account after several years.

(d) Find the minimum number of complete years when the amount of money in April's account exceeds that in Bea's account.

[3]

(e) Find the year when the sum of the amount of money in April's account and that in Bea's account first reaches $\$30500$.

[3]

4 Binomial Theorem

Important Notes

Properties of factorials and binomial coefficients:

1. $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$: n factorial
2. $0! = 1$
3. $C_r^n = \dfrac{n!}{r!(n-r)!}$: Binomial coefficient
4. $C_0^n = C_n^n = 1$
5. $C_1^n = C_{n-1}^n = n$
6. $C_r^n = C_{n-r}^n$ for $0 \le r \le n$, r, $n \in \mathbb{Z}^+$
7. $C_2^n = \dfrac{(n)(n-1)}{(2)(1)}$ can be expressed as a fraction in which both numerator and denominator are the product of two numbers, start descending from n and 2 respectively
8. $C_3^n = \dfrac{(n)(n-1)(n-2)}{(3)(2)(1)}$ can be expressed as a fraction in which both numerator and denominator are the product of three numbers, start descending from n and 3 respectively

Pascal triangle for finding binomial coefficients:

$n=0$	1
$n=1$	1 1
$n=2$	1 2 1
$n=3$	1 3 3 1
$n=4$	1 4 6 4 1
$n=5$	1 5 10 10 5 1
$\vdots$	$\vdots$
n	$1 \quad C_1^n \quad C_2^n \quad \cdots \quad C_{n-2}^n \quad C_{n-1}^n \quad 1$

Properties of binomial theorem:

1. $(a+b)^n = a^n + C_1^n a^{n-1} b^1 + C_2^n a^{n-2} b^2 + \cdots + C_r^n a^{n-r} b^r + \cdots + C_{n-1}^n a^1 b^{n-1} + b^n$: Binomial theorem for $0 \le r \le n$, r, $n \in \mathbb{Z}^+$
2. $C_r^n a^{n-r} b^r$: General term (Term in b^r)

Notes on GDC		
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Example 1.7

The binomial expansion of $(1+px)^n$ is $1+24x+qx^2+\cdots+4096x^6$, $n \in \mathbb{Z}^+$, p, $q \in \mathbb{R}$. Find the values of n, p and q.

[7]

$(1+px)^n$

$=1^n + C_1^n 1^{n-1}(px)^1 + C_2^n 1^{n-2}(px)^2 + \cdots + (px)^n$ — Binomial theorem (M1)

$=1+(n)(1)(px)+\left(\frac{(n)(n-1)}{(2)(1)}\right)(1)(p^2x^2)+\cdots+p^nx^n$ — $C_1^n = n$ & $C_2^n = \frac{n(n-1)}{2}$ (A1)

$=1+npx+\frac{n(n-1)p^2}{2}x^2+\cdots+p^nx^n$

The term of the largest power of x is $4096x^6$.

$\therefore n=6$ — (A1)

$npx = 24x$

$\therefore 6px = 24x$ — Correct equation (A1)

$p=4$ — (A1)

$\frac{n(n-1)p^2}{2}x^2 = qx^2$

$\therefore \frac{6(6-1)4^2}{2}x^2 = qx^2$ — Correct equation (A1)

$q=240$ — (A1)

Exercise 1.7

The binomial expansion of $(1+px)^n$ is $1+\frac{10}{3}x+\frac{40}{9}x^2+qx^3+\cdots+p^n x^n$, $n \in \mathbb{Z}^+$, p, $q \in \mathbb{R}$. Find the values of n, p and q.

[7]

Example 1.8

Consider the expansion of $x^3(2+x^2)^n$, $n \in \mathbb{Z}^+$. The coefficient of x^5 is 1024.

(a) Find n.

[6]

The general term

$= x^3 \cdot C_r^n (2)^{n-r} (x^2)^r$ — $C_r^n a^{n-r} b^r$ (M1)

$= x^3 \cdot C_r^n 2^{n-r} x^{2r}$

$= C_r^n 2^{n-r} x^{3+2r}$ — Term in x^{3+2r} (A1)

Consider the term in x^5.

$\therefore 3+2r=5$ — Correct equation (A1)

$2r=2$

$r=1$ — $r=1$ (A1)

The coefficient of x^5 is 1024

$\therefore C_1^n 2^{n-1} = 1024$ — Correct equation (A1)

$n \cdot 2^{n-1} = 1024$

$n \cdot 2^{n-1} - 1024 = 0$

By considering the graph of $y = n \cdot 2^{n-1} - 1024$,

the horizontal intercept is 8.

$\therefore n=8$ — (A1)

(b) Hence, find the coefficient of x^7.

[2]

$x^3(2+x^2)^8$

$= x^3\left[\cdots + C_2^8 2^{8-2}(x^2)^2 + \cdots\right]$ — Binomial theorem (M1)

$= x^3\left[\cdots + 1792x^4 + \cdots\right]$

$= \cdots + 1792x^7 + \cdots$

Thus, the coefficient of x^7 is 1792. — (A1)

Exercise 1.8

Consider the expansion of $\frac{(1+x^3)^n}{2x^2}$, $n \in \mathbb{Z}^+$. The coefficient of x^4 is 3.

(a) Find n.

[6]

(b) Hence, find the coefficient of x^7.

[2]

5 Proofs and Identities

Important Notes

Identity of x: The equivalence of two expressions on two sides of the identity sign $\equiv$, for all values of x

Useful identities and expressions for proofs:

1. $(a+b)^2 \equiv a^2 + 2ab + b^2$
2. $(a-b)^2 \equiv a^2 - 2ab + b^2$
3. $(a+b)(a-b) \equiv a^2 - b^2$
4. x & $x+1$: Two consecutive integers, where $x \in \mathbb{Z}$
5. $2x+1$ & $2x+3$: Two consecutive odd numbers, where $x \in \mathbb{Z}$
6. $2x$ & $2x+2$: Two consecutive even numbers, where $x \in \mathbb{Z}$
7. aN: A multiple of a, where a, $N \in \mathbb{Z}$

Example 1.9

(a) Show that $(5n)^2 + (5n+5)^2 = 50n^2 + 50n + 25$, $n \in \mathbb{Z}$.

[2]

L.H.S.

$= (5n)^2 + (5n+5)^2$ Starts from L.H.S. (M1)

$= 25n^2 + 25n^2 + 50n + 25$ $(a+b)^2 \equiv a^2 + 2ab + b^2$ (A1)

$= 50n^2 + 50n + 25$

$=$R.H.S.

$\therefore (5n)^2 + (5n+5)^2 = 50n^2 + 50n + 25$ (AG)

(b) Hence, or otherwise, prove that the sum of the squares of any two consecutive multiples of 5 is odd.

[3]

$5n$ and $5n+5$ are consecutive multiples of 5. Consecutive multiples (R1)

$(5n)^2 + (5n+5)^2 = 50n^2 + 50n + 25$ Proved in (a) (A1)

Also, $50n^2 + 50n + 25$ is an odd integer. $50n^2 + 50n$ is even (R1)

Thus, the sum of the squares of any two consecutive multiples of 5 is odd. (AG)

Exercise 1.9

(a) Show that $(3n)^2 + (3n+3)^2 = 18n^2 + 18n + 9$, $n \in \mathbb{Z}$.

[2]

(b) Hence, or otherwise, prove that the sum of the squares of any two consecutive multiples of 3 is divisible by 9.

[3]