Chapter 15 Solution

Exercise 53

1. (a) $f'(x)=3(4x^3)+\frac{1}{2}(1)$ (A1) for correct derivatives

$f'(x)=12x^3+\frac{1}{2}$ A1 N2

[2]

(b) The gradient

$=12(0)^3+\frac{1}{2}$ (A1) for substitution

$=\frac{1}{2}$ A1 N2

[2]

(c) The gradient of the normal

$=\frac{-1}{12(-1)^3+\frac{1}{2}}$ (A1) for substitution

$=\frac{2}{23}$ A1 N2

[2]

2. (a) $f'(x)=4(2x)-0$ (A1) for correct derivatives

$f'(x)=8x$ A1 N2

[2]

(b) The gradient

$=8\left(\frac{1}{4}\right)$ (A1) for substitution

$=2$ A1 N2

[2]

(c) The gradient of the normal is $\frac{-1}{8a}$. (M1) for valid approach

$\frac{-1}{8a}=\frac{1}{16}$ (A1) for correct equation

$-16=8a$

$a=-2$ A1 N3

[3]

3. (a) $f(-2) = \frac{1}{2}(-2)^2 - \frac{1}{-2}$ (A1) for substitution

$f(-2) = \frac{5}{2}$ A1 N2

[2]

(b) $f'(x) = \frac{1}{2}(2x) - (-x^{-2})$ (A1) for correct derivatives

$f'(x) = x + \frac{1}{x^2}$ A1 N2

[2]

(c) The gradient of the normal is $\frac{-1}{a + \frac{1}{a^2}}$. (M1) for valid approach

$\frac{-1}{a + \frac{1}{a^2}} = -\frac{9}{28}$ (A1) for correct equation

$28 = 9\left(a + \frac{1}{a^2}\right)$

$9a + \frac{9}{a^2} - 28 = 0$

By considering the graph of $y = 9a + \frac{9}{a^2} - 28$,

$a = -0.524461$ (*Rejected*),

$a = 0.6355726$ (*Rejected*) or $a = 3$. (A1) for correct values

$\therefore a = 3$ A1 N4

[4]

4. (a) 0 A1 N1

[1]

(b) $f'(x)=\frac{1}{a}(3x^2)+1$ (A1) for correct derivatives

$f'(x)=\frac{3}{a}x^2+1$ A1 N2

[2]

(c) The gradient of the normal is $\frac{-1}{\frac{3}{a}(2)^2+1}$. (M1) for valid approach

$\frac{-1}{\frac{3}{a}(2)^2+1}=-\frac{1}{3}$ (A1) for correct equation

$\frac{3}{a}(4)+1=3$

$\frac{12}{a}=2$

$a=6$ A1 N3

[3]

Exercise 54

1. (a) $f'(x)=6(4x^3)-21(2x)$ (A1) for correct derivatives

$f'(x)=24x^3-42x$ A1 N2

[2]

(b) The gradient of L

$=24(2)^3-42(2)$ (A1) for substitution

$=108$ A1 N2

[2]

(c) The equation of L:

$y-12=108(x-2)$ (A1) for substitution

$y-12=108x-216$

$y=108x-204$ A1 N2

[2]

2. (a) $f'(x)=3(1)-4(-2x^{-3})$ (A1) for correct derivatives

$f'(x)=3+8x^{-3}$ A1 N2

[2]

(b) The gradient of L

$=\frac{-1}{3+8(1)^{-3}}$ (A1) for substitution

$=-\frac{1}{11}$ A1 N2

[2]

(c) The equation of L:

$y-(-1)=-\frac{1}{11}(x-1)$ (A1) for substitution

$y+1=-\frac{1}{11}x+\frac{1}{11}$

$y=-\frac{1}{11}x-\frac{10}{11}$ A1 N2

[2]

3. (a) $f'(x) = a(3x^2) - 2(2x) + 0$ (A1) for correct derivatives

$f'(x) = 3ax^2 - 4x$ A1 N2

[2]

(b) $3a(3)^2 - 4(3) = 96$ (A1) for substitution

$27a = 108$

$a = 4$ A1 N2

[2]

(c) The equation of L:

$y - (27(4) - 17) = 96(x - 3)$ (A1) for substitution

$y - 91 = 96x - 288$

$y = 96x - 197$ A1 N2

[2]

4. (a) $f'(x) = 0 - a(3x^2)$ (A1) for correct derivatives

$f'(x) = -3ax^2$ A1 N2

[2]

(b) The gradient of L

$= \dfrac{-1}{-3a(2)^2}$ (A1) for substitution

$= \dfrac{1}{12a}$ A1 N2

[2]

(c) -1 A1 N1

[1]

(d) The equation of L:

$y - 11 = -\dfrac{1}{12}(x - 2)$ (A1) for substitution

$y - 11 = -\dfrac{1}{12}x + \dfrac{1}{6}$

$y = -\dfrac{1}{12}x + \dfrac{67}{6}$ A1 N2

[2]

Exercise 55

1. (a) $f'(x) = 2(2x) - 1$ (A1) for correct derivatives

$f'(x) = 4x - 1$ A1 N2

[2]

(b) $g'(x) = 2x$ A1 N1

[1]

(c) $f'(x) = g'(x)$

$4x - 1 = 2x$ (M1) for setting equation

$-1 = -2x$

$x = \frac{1}{2}$ A1 N2

[2]

(d) 1 A1 N1

[1]

2. (a) $f'(x) = 3x^2 + 2x + 0$ (A1) for correct derivatives

$f'(x) = 3x^2 + 2x$ A1 N2

[2]

(b) $g'(x) = 1$ A1 N1

[1]

(c) $f'(x) = g'(x)$

$3x^2 + 2x = 1$ (M1) for setting equation

$3x^2 + 2x - 1 = 0$

$(x+1)(3x-1) = 0$

$x = -1$ or $x = \frac{1}{3}$ A2 N3

[3]

(d) $x = -\frac{1}{3}$ A1 N1

[1]

3. (a) $f'(x) = 24(1) - 3x^2$ (A1) for correct derivatives

$f'(x) = 24 - 3x^2$ A1 N2

[2]

(b) $g'(x) = 3x^2$ A1 N1

[1]

(c) $f'(x) = g'(x)$

$24 - 3x^2 = 3x^2$ (M1) for setting equation

$24 = 6x^2$

$x^2 = 4$

$x = -2$ or $x = 2$ A2 N3

[3]

(d) The gradient of AB

$= \frac{(-2)^3 - 2^3}{-2-2}$ (A1) for substitution

$= \frac{-16}{-4}$

$= 4$ A1 N2

[2]

4. (a) (i) $f'(x) = -2ax$ A1 N1

(ii) $g'(x) = -4$ A1 N1

[2]

(b) $f'(2) = g'(2)$

$-2a(2) = -4$ (M1) for setting equation

$-4a = -4$

$a = 1$ A1 N2

[2]

(c) 1 A1 N1

[1]

(d) $\frac{0-1}{x-2} = -4$ (A1) for substitution

$\frac{-1}{x-2} = -4$

$x - 2 = \frac{1}{4}$

$x = \frac{9}{4}$

Thus, the x-intercept is $\frac{9}{4}$. A1 N2

[2]

Exercise 56

1.	(a)	$f'(x) = 2(3x^2) - 33(2x) + 108(1) - 0$	(A2) for correct derivatives
		$f'(x) = 6x^2 - 66x + 108$	A1 N3
			[3]
	(b)	$f'(x) < 0$	
		$6x^2 - 66x + 108 < 0$	(A1) for correct inequality
		$x^2 - 11x + 18 < 0$	
		By considering the graph of $y = x^2 - 11x + 18$,	
		$2 < x < 9$.	A2 N3
			[3]
2.	(a)	$f'(x) = -4x^3 + 20(3x^2) - 142(2x) + 420(1) + 0$	(A2) for correct derivatives
		$f'(x) = -4x^3 + 60x^2 - 284x + 420$	A1 N3
			[3]
	(b)	$f'(x) > 0$	
		$-4x^3 + 60x^2 - 284x + 420 > 0$	(A1) for correct inequality
		By considering the graph of	
		$y = -4x^3 + 60x^2 - 284x + 420$,	
		$x < 3$ or $5 < x < 7$.	A2 N3
			[3]
3.	(a)	(i) $y = 5$	A1 N1
		(ii) $y = 2$	A1 N1
			[2]
	(b)	$x < 3$ or $x > 11$	A2 N2
			[2]
	(c)	(11, 2)	A2 N2
			[2]
	(d)	$f(9)$	A1 N1
			[1]

4.	(a)	(i)	$y = 0$	A1	N1	
		(ii)	$y = 12$	A1	N1	
						[2]
	(b)		$x < -4$, $1 < x < 5$ or $x > 5$	A2	N2	
						[2]
	(c)		$(1, 20)$	A2	N2	
						[2]
	(d)		$f(2)$	A1	N1	
						[1]

Exercise 57

1.	(a)	$C'(x)=2x+0+54(-x^{-2})$	(A1) for correct derivative	
		$C'(x)=2x-\dfrac{54}{x^2}$	A1 N2	
				[2]
	(b)	$C'(x)=0$		
		$2x-\dfrac{54}{x^2}=0$	(M1) for setting equation	
		By considering the graph of		
		$y=2x-\dfrac{54}{x^2}$, $x=3$.	(M1) for valid approach	
		Thus, the required mass is 3 kg.	A1 N3	
				[3]
	(c)	$33	A1 N1	
				[1]
2.	(a)	1000	A1 N1	
				[1]
	(b)	$P'(t)=3t^2-12(2t)+36(1)+0$	(A1) for correct derivative	
		$P'(t)=3t^2-24t+36$	A1 N2	
				[2]
	(c)	$P'(t)=0$		
		$3t^2-24t+36=0$	(M1) for setting equation	
		By considering the graph of		
		$y=3t^2-24t+36$, $t=2$ or $t=6$ (*Rejected*).	(M1) for valid approach	
		Thus, the required number of days is 2.	A1 N3	
				[3]
	(d)	32001000	A1 N1	
				[1]

3. (a) $Q(t)=0$

$4t^2-120+\frac{216}{t}=0$ (M1) for setting equation

By considering the graph of $y=4t^2-120+\frac{216}{t}$,

$t=2.1156558$ or $t=4.1038579$.

Thus, $t=2.12$ or $t=4.10$. A2 N3

[3]

(b) $Q'(t)=4(2t)-0+216(-t^{-2})$ (A1) for correct derivative

$Q'(t)=8t-\frac{216}{t^2}$ A1 N2

[2]

(c) $Q'(t)=0$

$8t-\frac{216}{t^2}=0$ (M1) for setting equation

By considering the graph of $y=8t-\frac{216}{t^2}$, $t=3$. (M1) for valid approach

Thus, $t=3$. A1 N3

[3]

4. (a) $P'(t)=-3t^2+9(2t)-24(1)+0$ (A1) for correct derivative

$P'(t)=-3t^2+18t-24$ A1 N2

[2]

(b) $P'(t)=0$

$-3t^2+18t-24=0$ (M1) for setting equation

By considering the graph of $y=-3t^2+18t-24$,

$t=2$ or $t=4$ (*Rejected*). (M1) for valid approach

Thus, $t=2$. A1 N3

[3]

(c) The minimum price of the share is $700, which is greater than $690. A1 N1

[1]

Exercise 58

1.	(a)	$r = -(-2)^3 + 3(-2)^2 + 24(-2) - 1$	(A1) for substitution
		$r = -29$	A1 N2
			[2]
	(b)	$f'(x) = -3x^2 + 3(2x) + 24(1) - 0$	(A1) for correct derivatives
		$f'(x) = -3x^2 + 6x + 24$	A1 N2
			[2]
	(c)	$f'(x) = 0$	
		$-3x^2 + 6x + 24 = 0$	(M1) for setting equation
		$-3(x+2)(x-4) = 0$	
		$x = -2$ (*Rejected*) or $x = 4$	(A1) for correct value
		$f(4) = -4^3 + 3(4)^2 + 24(4) - 1$	
		$f(4) = 79$	
		Thus, the required coordinates are $(4, 79)$.	A1 N3
			[3]
	(d)	$x < -2$ or $x > 4$	A2 N2
			[2]
	(e)	(i) -1	A1 N1
		(ii) $f'(0) = -3(0)^2 + 6(0) + 24$	(M1) for substitution
		$f'(0) = 24$	A1 N2
		(iii) The equation of tangent:	
		$y - (-1) = 24(x - 0)$	(M1) for substitution
		$y + 1 = 24x$	
		$24x - y - 1 = 0$	A1 N2
		(iv) $24x - 0 - 1 = 0$	(M1) for substitution
		$24x = 1$	
		$x = \frac{1}{24}$	A1 N2
			[7]

2. (a) $350 = 2r^3 - 150r - 150$ (M1) for setting equation

$2r^3 - 150r - 500 = 0$

By considering the graph of $y = 2r^3 - 150r - 500$,

$r = -5$. A1 N2

[2]

(b) $f'(x) = 2(3x^2) - 150(1) - 0$ (A1) for correct derivatives

$f'(x) = 6x^2 - 150$ A1 N2

[2]

(c) $f'(x) = 0$

$6x^2 - 150 = 0$ (M1) for setting equation

$6(x+5)(x-5) = 0$

$x = -5$ (*Rejected*) or $x = 5$ (A1) for correct value

$f(5) = 2(5)^3 - 150(5) - 150$

$f(5) = -650$

Thus, the required coordinates are $(5, -650)$. A1 N3

[3]

(d) $x < -5$ or $x > 5$ A2 N2

[2]

(e) (i) -2 A1 N1

(ii) $f'(-1) = 6(-1)^2 - 150$ (M1) for substitution

$f'(-1) = -144$ A1 N2

(iii) $\dfrac{1}{144}$ A1 N1

(iv) The equation of normal:

$y - (-2) = \dfrac{1}{144}(x - (-1))$ (M1) for substitution

$y + 2 = \dfrac{1}{144}x + \dfrac{1}{144}$

$y = \dfrac{1}{144}x - \dfrac{287}{144}$ A1 N2

(v) $0 = \dfrac{1}{144}x - \dfrac{287}{144}$ (M1) for substitution

$0 = x - 287$

$x = 287$ A1 N2

[8]

3. (a) $x = 0$ A1 N1

[1]

(b) $f'(x) = 125(1) + 32(-2x^{-3})$ (A1) for correct derivatives

$f'(x) = 125 - \frac{64}{x^3}$ A1 N2

[2]

(c) $f'(x) = 0$

$125 - \frac{64}{x^3} = 0$ (M1) for setting equation

$125 = \frac{64}{x^3}$

$x^3 = \frac{64}{125}$

$x = 0.8$ (A1) for correct value

$f(0.8) = 125(0.8) + \frac{32}{0.8^2}$

$f(0.8) = 150$

Thus, the required coordinates are $(0.8, 150)$. A1 N3

[3]

(d) $0 < x < 0.8$ A2 N2

[2]

(e) (i) 157 A1 N1

(ii) $f'(1) = 125 - \frac{64}{1^3}$ (M1) for substitution

$f'(1) = 61$ A1 N2

(iii) The equation of tangent:

$y - 157 = 61(x - 1)$ (M1) for substitution

$y - 157 = 61x - 61$

$61x - y + 96 = 0$ A1 N2

[5]

(f) $y < 150$ A2 N2

[2]

4. (a) $x = 0$ A1 N1

[1]

(b) $f'(x) = \frac{1}{2}(2x) + 8(-2x^{-3})$ (A1) for correct derivatives

$f'(x) = x - \frac{16}{x^3}$ A1 N2

[2]

(c) $f'(x) = 0$

$x - \frac{16}{x^3} = 0$ (M1) for setting equation

$x = \frac{16}{x^3}$

$x^4 = 16$

$x = 2$ (A1) for correct value

$f(2) = \frac{1}{2}(2)^2 + \frac{8}{2^2}$

$f(2) = 4$

Thus, the required coordinates are $(2, 4)$. A1 N3

[3]

(d) $-1 < x < 0$ or $x > 2$ A2 N2

[2]

(e) (i) 8.5 A1 N1

(ii) $f'(4) = 4 - \frac{16}{4^3}$ (M1) for substitution

$f'(4) = \frac{15}{4}$ A1 N2

(iii) $-\frac{4}{15}$ A1 N1

(iv) The equation of normal:

$y - \frac{17}{2} = -\frac{4}{15}(x - 4)$ (M1) for substitution

$y - \frac{17}{2} = -\frac{4}{15}x + \frac{16}{15}$

$y = -\frac{4}{15}x + \frac{287}{30}$ A1 N2

[6]

(f) $y < 4$ A2 N2

[2]

Exercise 59

1.	(a)	(i)	$\pi r^2 + 2\pi rh = 27\pi$	(M1) for setting equation	
			$2\pi rh = 27\pi - \pi r^2$		
			$h = \dfrac{27 - r^2}{2r}$	A1	N2
		(ii)	$V = \pi r^2 h$		
			$V = \pi r^2 \left(\dfrac{27 - r^2}{2r} \right)$	A1	
			$V = \dfrac{\pi r}{2}(27 - r^2)$	A1	
			$V = \dfrac{27}{2}\pi r - \dfrac{1}{2}\pi r^3$	AG	N0

[4]

(b)	$\dfrac{dV}{dr} = \dfrac{27}{2}\pi(1) - \dfrac{1}{2}\pi(3r^2)$	(A1) for correct derivatives	
	$\dfrac{dV}{dr} = \dfrac{27}{2}\pi - \dfrac{3}{2}\pi r^2$	A1	N2

[2]

(c)	$\dfrac{dV}{dr} = 0$		
	$\dfrac{27}{2}\pi - \dfrac{3}{2}\pi r^2 = 0$	(M1) for setting equation	
	By considering the graph of $y = \dfrac{27}{2}\pi - \dfrac{3}{2}\pi r^2$,		
	$r = -3$ (*Rejected*) or $r = 3$.	(M1) for valid approach	
	Thus, the required radius is 3 cm.	A1	N3

[3]

(d)	(i)	The maximum volume		
		$= \dfrac{27}{2}\pi(3) - \dfrac{1}{2}\pi(3)^3$	(A1) for substitution	
		$= 27\pi \text{ cm}^3$	A1	N2
	(ii)	$0 \le V \le 27\pi$	A1	N1

[3]

(e) $V = 47$

$\frac{27}{2}\pi r - \frac{1}{2}\pi r^3 = 47$ (M1) for setting equation

$\frac{27}{2}\pi r - \frac{1}{2}\pi r^3 - 47 = 0$

By considering the graph of

$y = \frac{27}{2}\pi r - \frac{1}{2}\pi r^3 - 47$, $r = 1.1670634$ or

$r = 4.5133764$. (M1) for valid approach

Thus, the required radii are 1.17 cm and 4.51 cm. A1 N3

[3]

2.	(a)	(i)	$(64-2x)$ cm	A1 N1
		(ii)	$V = x(64-2x)^2$	A1
			$V = x(4096-256x+4x^2)$	A1
			$V = 4x^3-256x^2+4096x$	AG N0

[3]

(b) $\frac{dV}{dx} = 4(3x^2)-256(2x)+4096(1)$ (A1) for correct derivatives

$\frac{dV}{dx} = 12x^2-512x+4096$ A1 N2

[2]

(c) $\frac{dV}{dx} = 0$

$12x^2-512x+4096=0$ (M1) for setting equation

By considering the graph of

$y = 12x^2-512x+4096$, $x = \frac{32}{3}$ or

$x = 32$ (*Rejected*). (M1) for valid approach

Thus, $x = \frac{32}{3}$. A1 N3

[3]

(d) (i) The maximum volume

$= 4\left(\frac{32}{3}\right)^3 - 256\left(\frac{32}{3}\right)^2 + 4096\left(\frac{32}{3}\right)$ (A1) for substitution

$= 19418.07407$

$= 19418 \text{ cm}^3$ A1 N2

(ii) $0 \le V \le 19418$ A1 N1

[3]

(e) The total surface area

$= 2\left(64^2 - 4\left(\frac{32}{3}\right)^2\right)$ (M1)(A1) for substitution

$= 7281.777778$

$= 7280 \text{ cm}^2$ A1 N3

[3]

3. (a) (i) $100 = 2\left(\frac{1}{2}(r)(r)\sin 60^\circ\right) + 3rh$ A1

$100 = \frac{\sqrt{3}}{2}r^2 + 3rh$ A1

$200 = \sqrt{3}r^2 + 6rh$

$200 - \sqrt{3}r^2 = 6rh$ A1

$h = \frac{200 - \sqrt{3}r^2}{6r}$ AG N0

(ii) $V = \left(\frac{1}{2}(r)(r)\sin 60^\circ\right)h$ A1

$V = \left(\frac{\sqrt{3}}{4}r^2\right)\left(\frac{200 - \sqrt{3}r^2}{6r}\right)$

$V = \left(\frac{\sqrt{3}}{24}r\right)(200 - \sqrt{3}r^2)$ A1

$V = \frac{25\sqrt{3}}{3}r - \frac{1}{8}r^3$ AG N0

[5]

(b) $\frac{dV}{dr} = \frac{25\sqrt{3}}{3}(1) - \frac{1}{8}(3r^2)$ (A1) for correct derivatives

$\frac{dV}{dr} = \frac{25\sqrt{3}}{3} - \frac{3}{8}r^2$ A1 N2

[2]

(c) $\frac{dV}{dr} = 0$

$\frac{25\sqrt{3}}{3} - \frac{3}{8}r^2 = 0$ (M1) for setting equation

By considering the graph of $y = \frac{25\sqrt{3}}{3} - \frac{3}{8}r^2$,

$r = -6.2040324$ (*Rejected*) or $r = 6.2040324$. (M1) for valid approach

Thus, $r = 6.20$. A1 N3

[3]

(d) The maximum volume

$= \frac{25\sqrt{3}}{3}(6.2040324) - \frac{1}{8}(6.2040324)^3$ (A1) for substitution

$= 59.69832955$

$= 59.7\text{ cm}^3$ A1 N2

[2]

(e) The required density

$= \dfrac{9}{59.69832955}$ (M1) for valid approach

$= 0.1507579872$

$= 1.51 \times 10^{-1}$ kg/cm^3 A1 N2

[2]

4. (a) (i) $168=\left(\frac{\pi r^2}{4}\right)(h)$ A1

$h=\frac{672}{\pi r^2}$ A1 N2

(ii) $A=\left(\frac{\pi r^2}{4}\right)(2)+2rh+\left(\frac{2\pi r}{4}\right)(h)$ A1

$A=\frac{1}{2}\pi r^2+2r\left(\frac{672}{\pi r^2}\right)+\left(\frac{\pi r}{2}\right)\left(\frac{672}{\pi r^2}\right)$ A1

$A=\frac{1}{2}\pi r^2+\frac{1344}{\pi r}+\frac{336}{r}$ A1

$A=\frac{1}{2}\pi r^2+\left(\frac{1344}{\pi}+336\right)\frac{1}{r}$ AG N0

[5]

(b) $\frac{\mathrm{d}A}{\mathrm{d}r}=\frac{1}{2}\pi(2r)+\left(\frac{1344}{\pi}+336\right)(-r^{-2})$ (A1) for correct derivatives

$\frac{\mathrm{d}A}{\mathrm{d}r}=\pi r-\left(\frac{1344}{\pi}+336\right)\frac{1}{r^2}$ A1 N2

[2]

(c) $\frac{\mathrm{d}A}{\mathrm{d}r}=0$

$\pi r-\left(\frac{1344}{\pi}+336\right)\frac{1}{r^2}=0$ (M1) for setting equation

By considering the graph of

$y=\pi r-\left(\frac{1344}{\pi}+336\right)\frac{1}{r^2}$, $r=6.2413452$. (M1) for valid approach

Thus, $r=6.24$. A1 N3

[3]

(d) The minimum total surface area

$=\frac{1}{2}\pi(6.2413452)^2+\left(\frac{1344}{\pi}+336\right)\frac{1}{6.2413452}$ (A1) for substitution

$=183.5682368$

$=184\text{ cm}^2$ A1 N2

[2]

(e) The minimum number of painting buckets

$=\frac{183.5682368}{25}$ (M1) for valid approach

$=7.342729471$

Thus, 8 buckets are needed. A1 N2

[2]