SE Production

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# Chapter 2

# Quadratic Functions

**SUMMARY POINTs**

✓ General form $y = ax^2 + bx + c$, where $a \neq 0$:

| | |
|---|---|
| $a > 0$ | The graph opens upward |
| $a < 0$ | The graph opens downward |
| $c$ | $y$-intercept |
| $h = -\frac{b}{2a}$ | $x$-coordinate of the vertex |
| $k = ah^2 + bh + c$ | $y$-coordinate of the vertex |
| | Extreme value of $y$ |
| $x = h$ | Equation of the axis of symmetry |

✓ Other forms:

1. $y = a(x-h)^2 + k$ : Vertex form
2. $y = a(x-p)(x-q)$ : Intercept form with $x$-intercepts $p$ and $q$

## SUMMARY POINTs

✓ Solving the quadratic equation $ax^2+bx+c=0$, where $a\neq 0$:

1. Factorization by cross method
2. $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ : Quadratic Formula
3. Method of completing the square

✓ The discriminant $\Delta=b^2-4ac$ of $ax^2+bx+c=0$:

| | |
|---|---|
| $\Delta>0$ | The quadratic equation has two distinct real roots |
| $\Delta=0$ | The quadratic equation has one double real root |
| $\Delta<0$ | The quadratic equation has no real root |

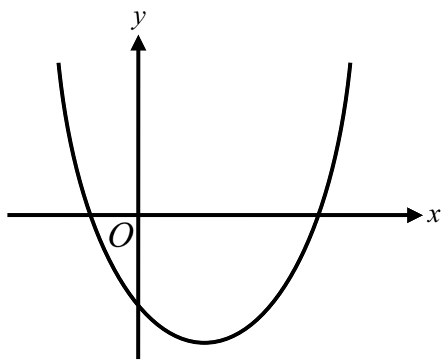
✓ $x$-intercepts of a quadratic function $y=ax^2+bx+c$: Roots of $ax^2+bx+c=0$

**Solutions of Chapter 2**

# 2 Paper 1 Section A – Find $x$-intercepts and the coordinates of the vertex in $y = ax^2 + bx + c$

**Example**

Let $f(x) = 3x^2 - 12x - 15$. Part of the graph of $f$ is shown below.




(a) Find the $x$-intercepts of the graph.

[4]

(b) (i) Write down the equation of the axis of symmetry.

(ii) Find the $y$-coordinate of the vertex.

[3]

**Solution**

(a) $f(x) = 0$ — (M1) for function equals to 0

$3x^2 - 12x - 15 = 0$

$3(x+1)(x-5) = 0$ — A1

$x = -1$ or $x = 5$

Thus, the $x$-intercepts are $-1$ and 5. — A2 N2

[4]

(b) (i) $x = 2$ — A1 N1

(ii) The $y$-coordinate of the vertex

$= 3(2)^2 - 12(2) - 15$ — (M1) for substitution

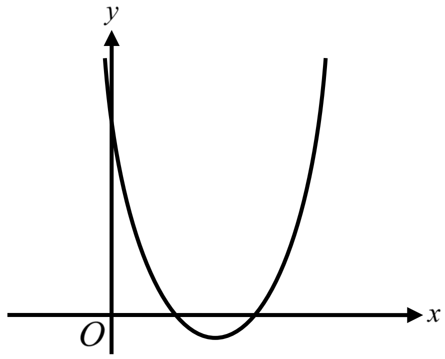
$= -27$ — A1 N2

[3]

**Exercise 2**



**1.** Let $f(x) = x^2 - 6x + 8$. Part of the graph of $f$ is shown below.




(a) Find the $x$-intercepts of the graph.

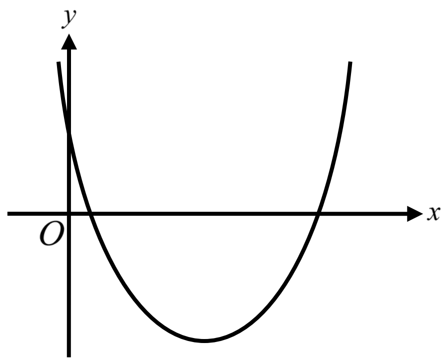
[4]

(b) (i) Write down the equation of the axis of symmetry.

(ii) Find the $y$-coordinate of the vertex.

[3]

**2.** Let $f(x) = x^2 - 11x + 10$. Part of the graph of $f$ is shown below.




(a) Find the $x$-intercepts of the graph.

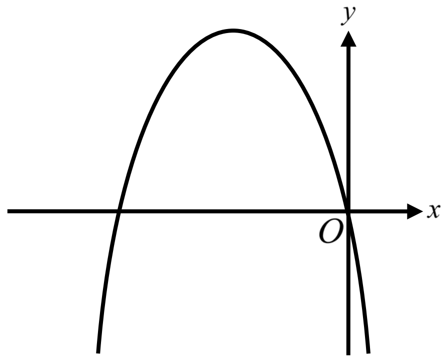
[4]

(b) (i) Write down the equation of the axis of symmetry.

(ii) Find the $y$-coordinate of the vertex.

[3]

**3.** Let $f(x) = -2x^2 - 14x$. Part of the graph of $f$ is shown below.




(a) Find the $x$-intercepts of the graph.

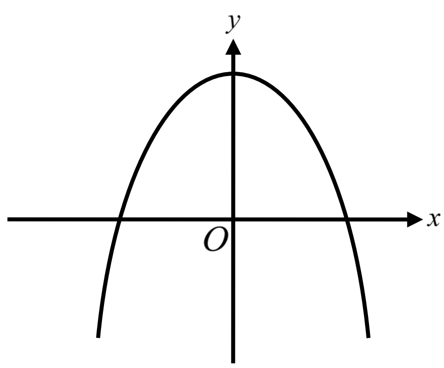
[4]

(b) (i) Write down the equation of the axis of symmetry.

(ii) Find the $y$-coordinate of the vertex.

[3]

**4.** Let $f(x) = 13.5 - 1.5x^2$. Part of the graph of $f$ is shown below.




(a) Find the $x$-intercepts of the graph.

[4]

(b) (i) Write down the equation of the axis of symmetry.

(ii) Find the $y$-coordinate of the vertex.

[3]

# 3 Paper 1 Section A – Find the coordinates of the vertex in $y = a(x-p)(x-q)$

## Example

Let $f(x) = (x-2)(x+4)$.

(a) Write down the $x$-intercepts of the graph of $f$.

[2]

(b) Find the coordinates of the vertex of the graph of $f$.

[4]

## Solution

(a) $x = 2$ and $x = -4$ A2 N2

[2]

(b) $h = \dfrac{2+(-4)}{2}$ (M1) for correct formula

$h = -1$ A1

$k = (-1-2)(-1+4)$ (M1) for finding $k$

$k = -9$

Thus, the coordinates of the vertex are $(-1, -9)$. A1 N3

[4]

## Exercise 3

**1.** Let $f(x) = (x-7)(x+5)$.

(a) Write down the $x$-intercepts of the graph of $f$.

[2]

(b) Find the coordinates of the vertex of the graph of $f$.

[4]

**2.** Let $f(x) = 2(x+1)(x+6)$.

(a) Write down the $x$-intercepts of the graph of $f$.

[2]

(b) Find the coordinates of the vertex of the graph of $f$.

[4]

**3.** Let $f(x) = a(x - p)(x - q)$.

The graph of $f(x)$ passes through the points $(5, 0)$, $(10, -7.5)$ and $(11, 0)$.

(a) Write down the value of $p$ and of $q$.

[2]

(b) Write down the **equation** of the axis of symmetry.

[1]

(c) Find the value of $a$.

[3]

**4.** Let $f(x) = a(p - x)(x - q)$.

The graph of $f(x)$ passes through the origin, $(15, 30)$ and $(18, 0)$.

(a) Write down the value of $p$ and of $q$.

[2]

(b) Write down the **equation** of the axis of symmetry.

[1]

(c) Find the value of $a$.

[3]

# 4 Paper 1 Section A – Find the unknown coefficients of a quadratic equation

## Example

The equation $kx^2+(k-3)x-3=0$ has two distinct real roots. Find the possible values of $k$.

[8]

## Solution

| | |
|---|---|
| $\Delta = b^2-4ac$ | (M1) for discriminant |
| $b^2-4ac>0$ | R1 |
| $(k-3)^2-4(k)(-3)>0$ | (A1) for substitution |
| $k^2-6k+9+12k>0$ | |
| $k^2+6k+9>0$ | A1 |
| $(k+3)^2>0$ | (M1) for factorizing |
| $k+3<0$ or $k+3>0$ | A1 |
| $k<-3$ or $k>-3$ | A2 N4 |

[8]

## Exercise 4

**1.** The equation $x^2-5x+k^2=0$ has two equal real roots. Find the values of $k$.

[7]

**2.** The equation $x^2+4kx+2k=0$ has two distinct real roots. Find the possible values of $k$.

[7]

**3.** The equation $x^2+1=(1-k)x$ has no real root. Find the possible values of $k$.

[8]

**4.** The equation $4x^2+(4k+16)x+25k=0$ has real roots. Find the possible values of $k$.

[8]

# 5 Paper 1 Section B – Find the unknown coefficients of a tangent

**Example**

A quadratic function $f$ can be written in the form $f(x) = a(x - p)(x - 7)$. The graph of $f$ has axis of symmetry $x = 3$ and $y$-intercept at $(0, -7)$.

(a) Find the value of $p$.

[3]

(b) Find the value of $a$.

[3]

(c) The line $y = mx - 11$ is a tangent to the curve of $f$. Find the values of $m$.

[8]

**Solution**

(a) $x = 7$ is one of the $x$-intercepts. (M1) for valid approach

$\frac{p+7}{2} = 3$ (M1) for correct formula

$p = -1$ A1 N2

[3]

(b) $-7 = a(0 - (-1))(0 - 7)$ (M1) for substitution

$-7 = -7a$ (A1) for simplification

$a = 1$ A1 N2

[3]

(c) A tangent only intersects with a curve once. (M1) for correct property

It implies that the corresponding discriminant equals to 0. R1

$(x - (-1))(x - 7) = mx - 11$ (M1) for setting equation

$x^2 - 6x - 7 = mx - 11$

$x^2 + (-6 - m)x + 4 = 0$ (M1) for quadratic equation

$(-6 - m)^2 - 4(1)(4) = 0$ A1

$36 + 12m + m^2 - 16 = 0$

$m^2 + 12m + 20 = 0$

$(m + 2)(m + 10) = 0$ (A1) for factorization

$m = -2$ or $m = -10$ A2 N0

[8]

## Exercise 5



**1.** A quadratic function $f$ can be written in the form $f(x) = a(x-p)(x+2)$. The graph of $f$ has axis of symmetry $x = 1$ and $y$-intercept at $(0, -32)$.

(a) Find the value of $p$.

[3]

(b) Find the value of $a$.

[3]

(c) The line $y = 4mx - 57$ is a tangent to the curve of $f$. Find the values of $m$.

[8]

**2.** A quadratic function $f$ can be written in the form $f(x) = a(x-4)(x-q)$. The graph of $f$ has axis of symmetry $x = 2.5$ and passes through $(5, -4)$.

(a) Find the value of $q$.

[3]

(b) Find the value of $a$.

[3]

(c) The line $y = mx$ is a tangent to the curve of $f$. Find the values of $m$.

[8]

**3.** A quadratic function $f$ can be written in the form $f(x) = (x-p)(x-1)$. The graph of $f$ passes through $(3, 12)$.

(a) Find the value of $p$.

[3]

(b) Find the $x$-coordinate of the vertex of $f$.

[3]

(c) The line $y = m(x-1)$ is a tangent to the curve of $f$. Find the values of $m$.

[8]

**4.** A quadratic function $f$ can be written in the form $f(x) = a(x-p)(x+p)$, where $p > 0$. The graph of $f$ passes through $(0, -9)$ and $(1, -5)$.

(a) Show that $a = \dfrac{9}{p^2}$.

[2]

(b) Hence, find the values of $p$ and $a$.

[4]

(c) The line $y = -4mx - (9+m)$ is a tangent to the curve of $f$. Find the values of $m$.

[8]

# 6 Paper 2 Section A – Find the unknown coefficients of a curve with given number of intersections

## Example

Let $f(x)=x^2+2kx$ and $g(x)=6x-1$. The graphs of $f$ and $g$ intersect at two distinct points. Find the possible values of $k$.

[8]

## Solution

| | |
|---|---|
| $x^2+2kx=6x-1$ | (M1) for setting equation |
| $x^2+(2k-6)x+1=0$ | M1 |
| $\Delta=b^2-4ac>0$ | (M1)R1 for discriminant |
| $(2k-6)^2-4(1)(1)>0$ | (A1) for substitution |
| $4k^2-24k+32>0$ | |
| $k^2-6k+8>0$ | |
| $(k-2)(k-4)>0$ | (M1) for factorization |
| $k<2$ or $k>4$ | A2 N3 |

[8]

## Exercise 6

**1.** Let $f(x)=-x^2-4x$ and $g(x)=2kx+1$. The graphs of $f$ and $g$ do not intersect with each other. Find the possible values of $k$.

[8]

**2.** Let $f(x)=x^2-4x-4k$ and $g(x)=2kx-16$. The graphs of $f$ and $g$ intersect at two distinct points. Find the possible values of $k$.

[8]

**3.** Let $f(x)=x^2-1.5k$ and $g(x)=-16+(8-k)x$. The graphs of $f$ and $g$ intersect with each other. Find the possible values of $k$.

[8]

**4.** Let $f(x)=x^2+2x-2k$ and $g(x)=9-kx$. The graphs of $f$ and $g$ intersect with each other at most once. Find the possible values of $k$.

[8]

# Chapter

# 3

# Functions

**SUMMARY POINTs**

✓ The function $y = f(x)$:

1. $f(a)$: Functional value when $x = a$
2. Domain: Set of values of $x$
3. Range: Set of values of $y$

✓ $f \circ g(x) = f(g(x))$: Composite function of $f(x)$ with $g(x)$

✓ Steps of finding the inverse function $y = f^{-1}(x)$ of $f(x)$:

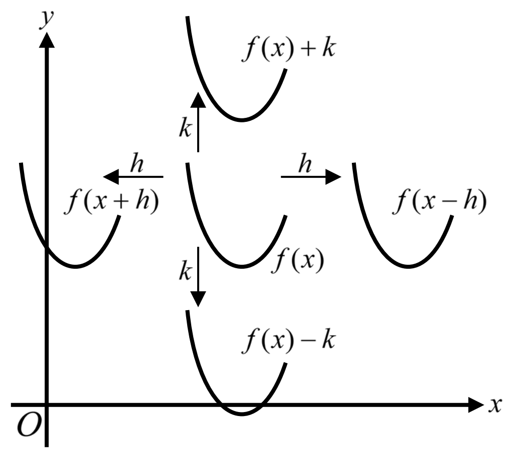
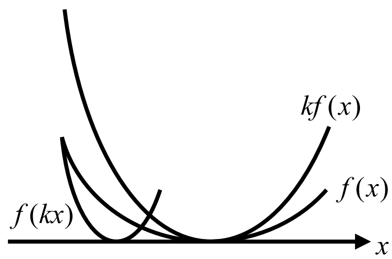
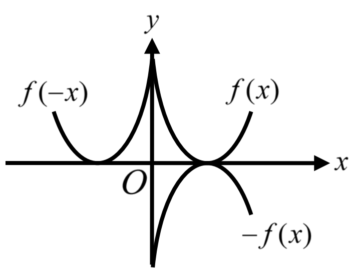
1. Start from stating $y$ in terms of $x$
2. Interchange $x$ and $y$
3. Make $y$ the subject in terms of $x$

✓ Properties of $y = f^{-1}(x)$:

1. $f(f^{-1}(x)) = f^{-1}(f(x)) = x$
2. The graph of $y = f^{-1}(x)$: Reflection of the graph of $y = f(x)$ about $y = x$

## *SUMMARY POINTs*

✓ Summary of transformations:

| | |
|---|---|
|  | $f(x) \to f(x)+k$:<br>Translate upward by $k$ units |
| | $f(x) \to f(x)-k$:<br>Translate downward by $k$ units |
| | $f(x) \to f(x+h)$:<br>Translate to the left by $h$ units |
| | $f(x) \to f(x-h)$:<br>Translate to the right by $h$ units |
|  | $f(x) \to kf(x)$:<br>Vertical stretch of scale factor $k$ |
| | $f(x) \to f(kx)$:<br>Horizontal compression of scale factor $k$ |
|  | $f(x) \to -f(x)$:<br>Reflection about the $x$-axis |
| | $f(x) \to f(-x)$:<br>Reflection about the $y$-axis |

✓ Properties of rational function $y=\dfrac{ax+b}{cx+d}$:

1. $y=\dfrac{1}{x}$: Reciprocal function
2. $y=\dfrac{a}{c}$: Horizontal asymptote
3. $x=-\dfrac{d}{c}$: Vertical asymptote

**Solutions of Chapter 3**

## 7 Paper 1 Section A – Find the inverse function and the composite of two functions



**Example**

Let $f(x)=3x+4$ and $g(x)=7x^2-1$.

(a) Find $f^{-1}(x)$.

[3]

(b) Find $(g\circ f)(2)$.

[3]

**Solution**

(a) $y=3x+4$

$\Rightarrow x=3y+4$ (M1) for swapping variables

$x-4=3y$

$y=\dfrac{x-4}{3}$ (A1) for changing subject

$\therefore f^{-1}(x)=\dfrac{x-4}{3}$ A1 N2

[3]

(b) $f(2)$

$=3(2)+4$ (M1) for substitution

$=10$

$(g\circ f)(2)$

$=g(10)$

$=7(10)^2-1$ (A1) for substitution

$=699$ A1 N3

[3]

**Exercise 7**

**1.** Let $f(x)=8x-1$ and $g(x)=x^2-5$.

(a) Find $f^{-1}(x)$.

[3]

(b) Find $(f\circ g)(5)$.

[3]

**2.** Let $f(x) = 2x - 3$ and $g(x) = (x+5)^2$.

(a) Find $f^{-1}(x)$.

[3]

(b) Find $(g \circ f)(-2)$.

[3]

**3.** Let $f(x) = \sqrt{x+4}$, for $x \geq -4$.

(a) Find $f^{-1}(4)$.

[3]

(b) Let $g$ be a function such that $g^{-1}$ exists for all real numbers. Given that $g(96) = 7$, find $(f \circ g^{-1})(7)$.

[3]

**4.** Let $f(x) = \sqrt{2x-1}$, for $x \geq \frac{1}{2}$.

(a) Find $f^{-1}(3)$.

[3]

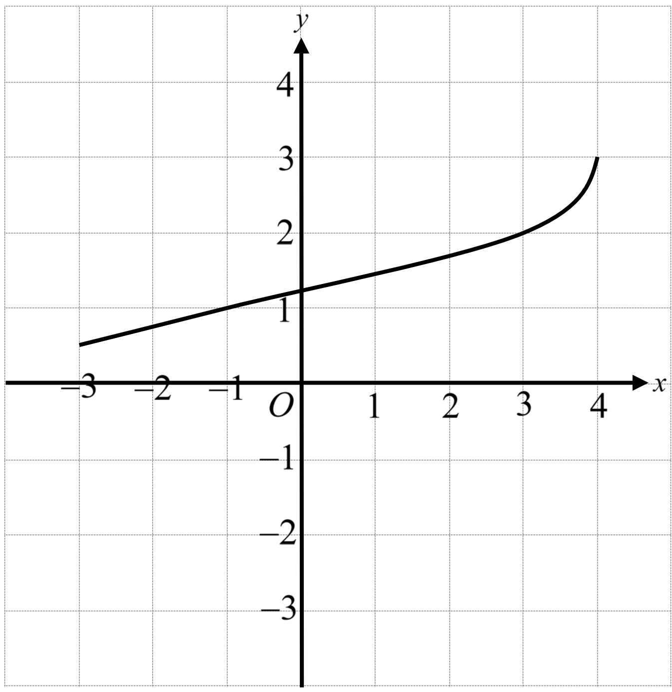
(b) Let $g$ be a function such that $g^{-1}$ exists for all real numbers. Given that $g\left(\frac{3a+1}{2}\right) = 2$, where $a$ is a constant, find $(f \circ g^{-1})(2)$, give the answer in terms of $a$.

[3]

# 8 Paper 1 Section A – Reflection of the graph about $x$ or $y$-axis

**Example**

The following diagram shows the graph of a function $f$.




(a) Find $f^{-1}(2)$.

[2]

(b) Find $(f \circ f)(4)$.

[3]

(c) On the same diagram, sketch the graph of $y = -f(x)$.

[2]

**Solution**

(a) $f(3)=2$ — (M1) for correct approach

$\therefore f^{-1}(2)=3$ — A1 N2

[2]

(b) $f(4)=3$ — (M1) for correct approach

$(f\circ f)(4)$

$=f(3)$ — (A1) for composite function

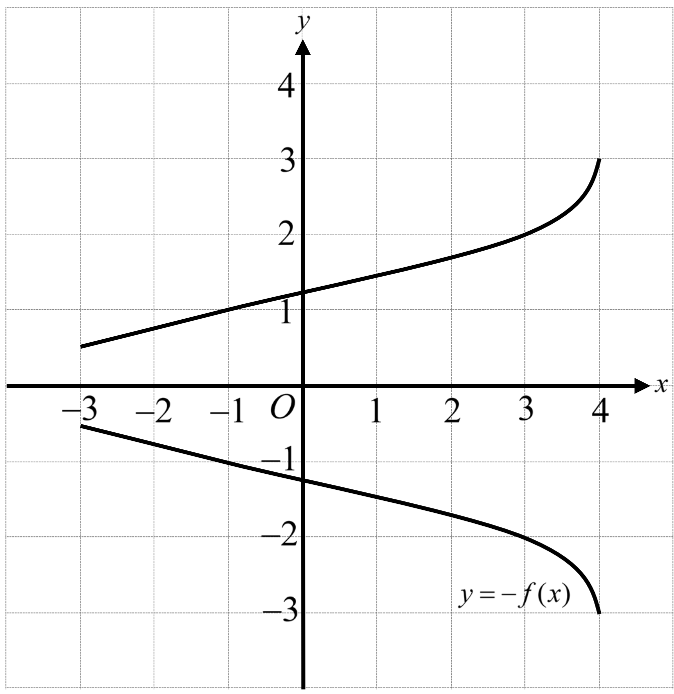
$=2$ — A1 N3

[3]

(c) For correct $y$-intercept — A1

For any two correct points from $(-1,-1)$, $(3,-2)$ and $(4,-3)$ — A1 N2

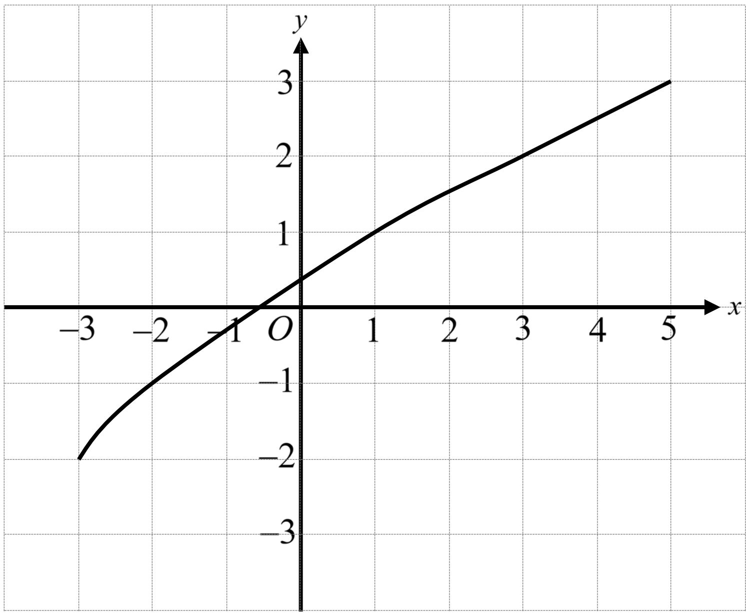
[2]



**Exercise 8**



**1.** The following diagram shows the graph of a function $f$.

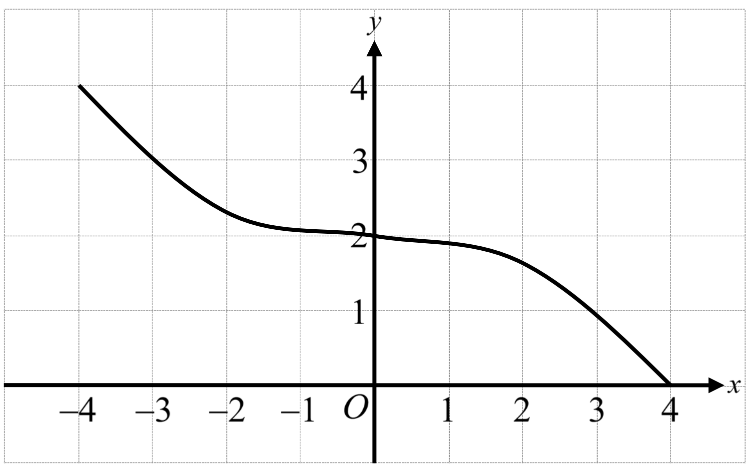



(a) Find $f^{-1}(-2)$. [2]

(b) Find $(f \circ f)(5)$. [3]

(c) On the same diagram, sketch the graph of $y = -f(x)$. [2]

**2.** The following diagram shows the graph of a function $f$.

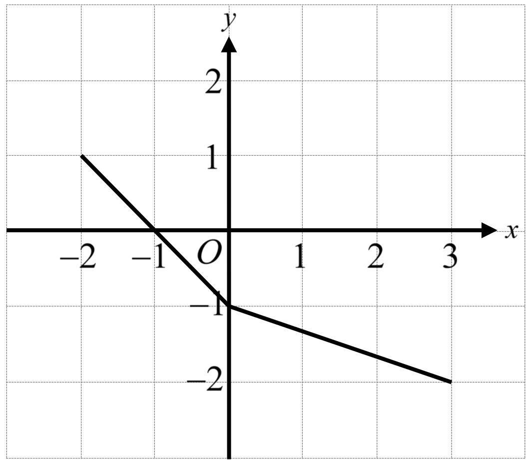



(a) Find $f^{-1}(2)$. [2]

(b) Find $(f \circ f)(4)$. [3]

(c) On the same diagram, sketch the graph of $y = f(-x)$. [2]

**3.** The following diagram shows the graph of a function $f$.




(a) Find the range of $f^{-1}$.

[2]

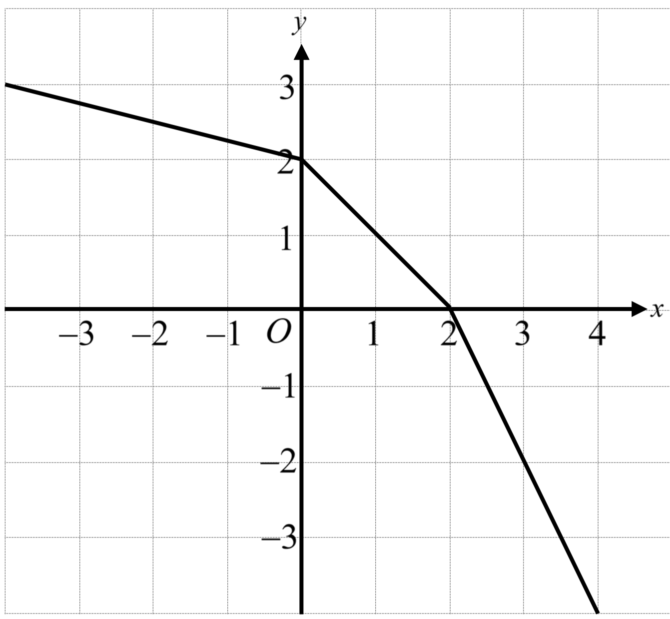
(b) Find $(f^{-1} \circ f^{-1})(1)$.

[3]

(c) On the same diagram, sketch the graph of $y = -f(x)$.

[2]

**4.** The following diagram shows the graph of a function $f$.




(a) Find the domain of $f^{-1}$.

[2]

(b) Find $(f^{-1} \circ f^{-1})(3)$.

[3]

(c) On the same diagram, sketch the graph of $y = f(-x)$.

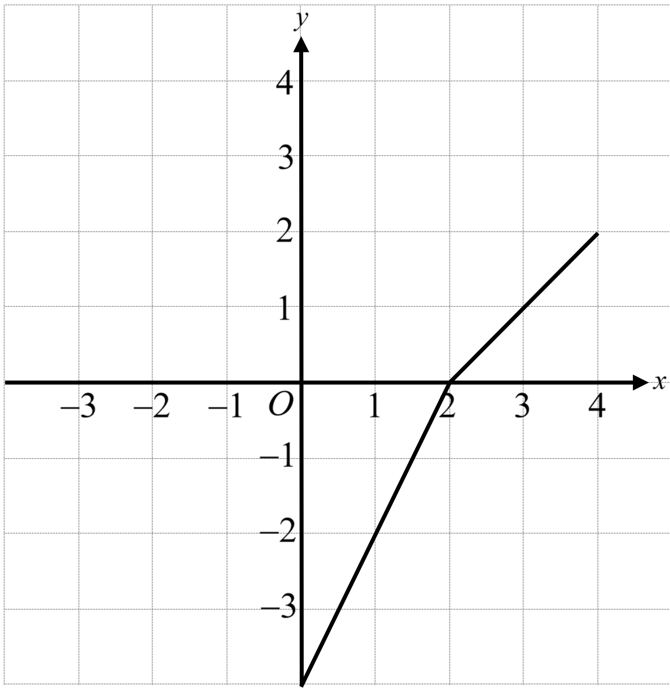
[2]

## 9 Paper 1 Section A – Sketching of the inverse graph

**Example**



The diagram below shows the graph of a function $f$, for $0 \le x \le 4$.




(a) Write down the value of

(i) $f(1)$;

(ii) $f^{-1}(2)$.

[3]

(b) On the same diagram, sketch the graph of $f^{-1}$.

[3]

**Solution**

(a) (i) $f(1) = -2$ A1 N1

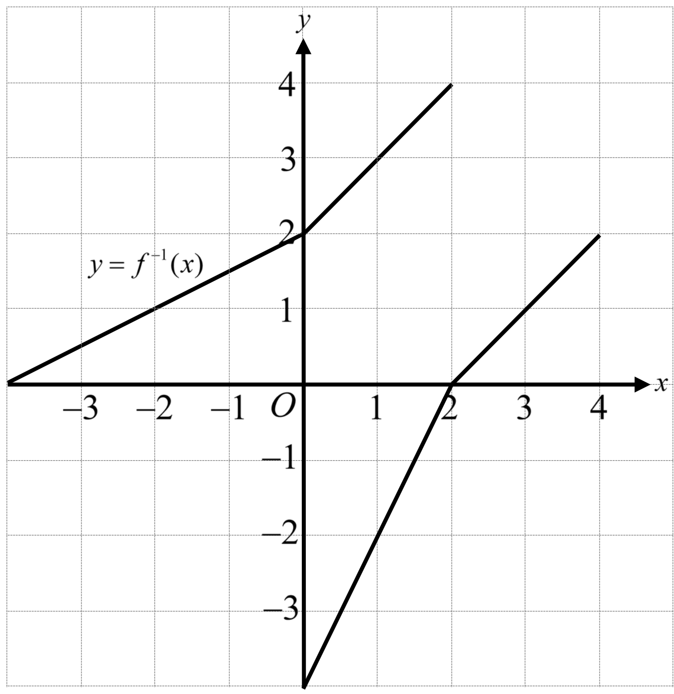
(ii) $f^{-1}(2) = 4$ A2 N2

[3]

(b) For any two correct points from $(-4, 0)$, $(0, 2)$ and $(2, 4)$ M1

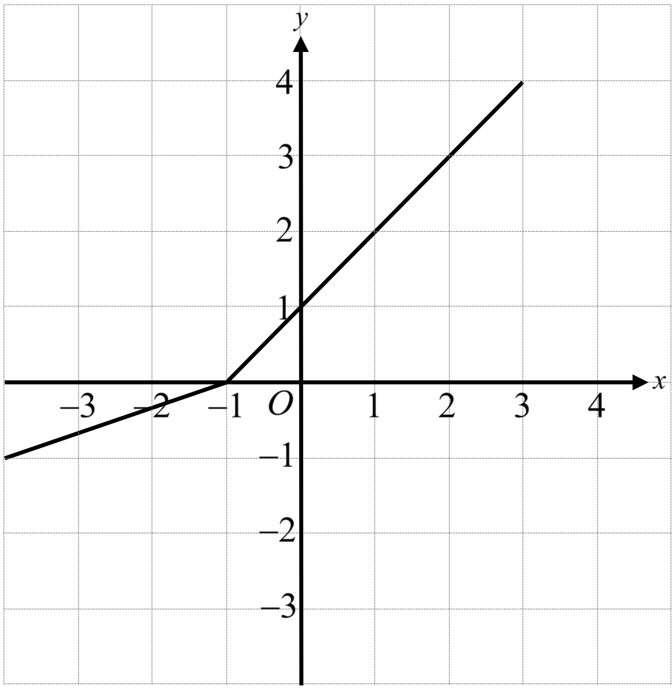
For correct graph A2 N3

[3]



Exercise 9

1. The diagram below shows the graph of a function $f$, for $-4 \le x \le 3$.




(a) Write down the value of

(i) $f(2)$;

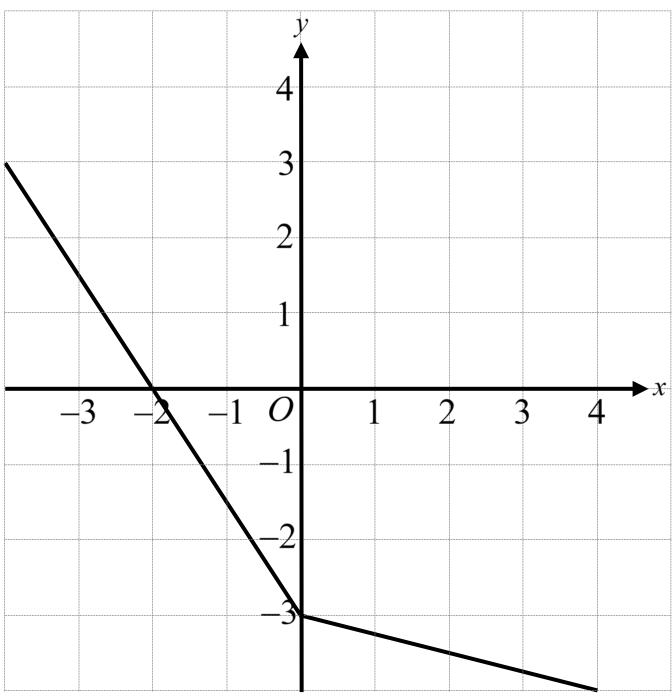
(ii) $f^{-1}(-1)$.

[3]

(b) On the same diagram, sketch the graph of $f^{-1}$.

[3]

2. The diagram below shows the graph of a function $f$, for $-4 \le x \le 4$.



(a) Write down the value of

(i) $f(-4)$;

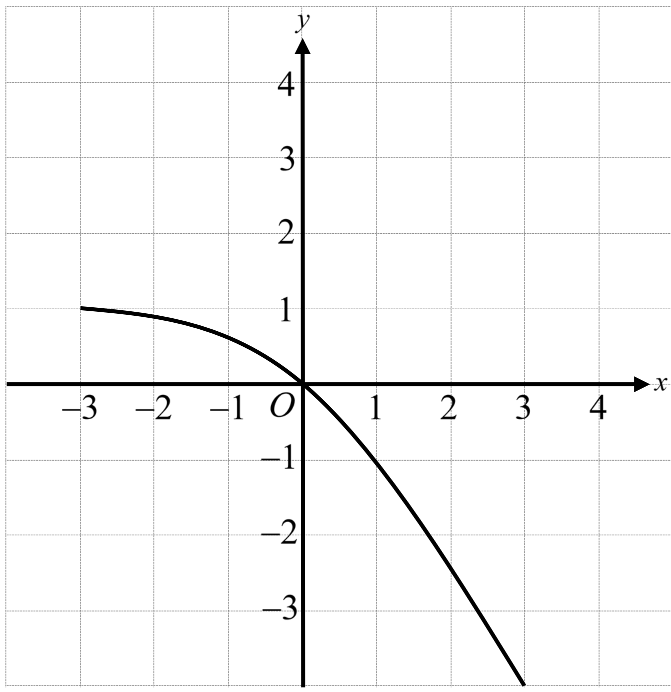
(ii) $f^{-1}(-4)$.

[3]

(b) On the same diagram, sketch the graph of $f^{-1}$.

[3]

**3.** The diagram below shows the graph of a function $f$, for $-3 \le x \le 3$.



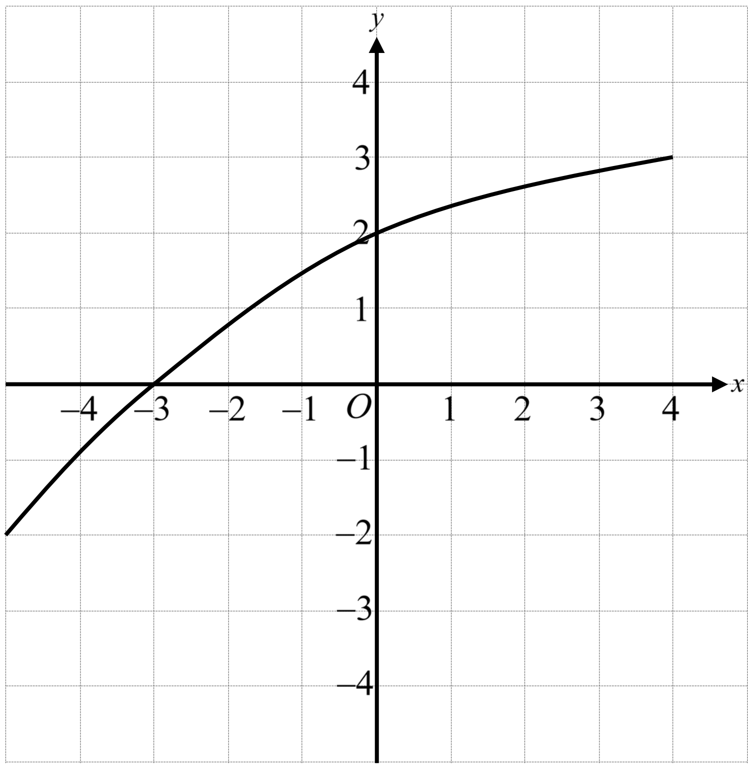

(a) On the same diagram, sketch the graph of $f^{-1}$.

[3]

(b) Let $g(x) = 2f(x+1)$. The point $\text{A}(1, -1)$ on the graph of $f$ is transformed to the point B on the graph of $g$. Find the coordinates of B.

[3]

**4.** The diagram below shows the graph of a function $f$, for $-5 \le x \le 4$.




(a) On the same diagram, sketch the graph of $f^{-1}$.

[3]

(b) Let $g(x) = f(2x) - 3$. The point $\mathrm{A}(-5, -2)$ on the graph of $f$ is transformed to the point B on the graph of $g$. Find the coordinates of B.

[3]

3

# 10 Paper 1 Section B – Transformations in quadratic functions

## Example

A quadratic function $f$ is given by $f(x)=(x-h)^2+k$.

The vertex of the graph of $f$ is at $(-3,-1)$, and the graph crosses the $y$-axis at the point $(0,c)$.

(a) Write down the value of $h$ and of $k$.

[2]

(b) Find the value of $c$.

[2]

Let $g(x)=-(x+5)^2+19$. The graph of $g$ is obtained by a reflection of the graph of $f$ in the $x$-axis, followed by a translation of $\begin{pmatrix} p \\ q \end{pmatrix}$.

(c) Find the value of $p$ and of $q$.

[5]

(d) Find the $x$-coordinates of the points of intersection of the graphs of $f$ and $g$.

[7]

## Solution

(a) $h=-3$, $k=-1$ A2 N2

[2]

(b) $f(x)=(x+3)^2-1$

$c$

$=f(0)$ (M1) for substitution

$=(0+3)^2-1$

$=8$ A1 N2

[2]

(c) $g(x)=-\left[((x-p)+3)^2-1\right]+q$ A1

$g(x)=-(x+(-p+3))^2+(1+q)$

$-p+3=5$ (M1) for translation

$p=-2$ A1 N2

$1+q=19$ (M1) for translation

$q=18$ A1 N2

[5]

(d) $f(x) = g(x)$

| | |
|---|---|
| $(x+3)^2 - 1 = -(x+5)^2 + 19$ | M1 |
| $x^2 + 6x + 9 - 1 = -x^2 - 10x - 25 + 19$ | (A1) for expansion |
| $x^2 + 6x + 8 = -x^2 - 10x - 6$ | (A1) for simplification |
| $2x^2 + 16x + 14 = 0$ | A1 |
| $x^2 + 8x + 7 = 0$ | |
| $(x+7)(x+1) = 0$ | (A1) for factorization |
| $x = -7$ or $x = -1$ | A2 N3 |

[7]

## Exercise 10

**1.** A quadratic function $f$ is given by $f(x) = -(x-h)^2 + k$.

The vertex of the graph of $f$ is at $(3, -1)$, and the graph crosses the $y$-axis at the point $(0, c)$.

(a) Write down the value of $h$ and of $k$.

[2]

(b) Find the value of $c$.

[2]

Let $g(x) = (x-1)^2 - 5$. The graph of $g$ is obtained by a reflection of the graph of $f$ in the $x$-axis, followed by a translation of $\begin{pmatrix} p \\ q \end{pmatrix}$.

(c) Find the value of $p$ and of $q$.

[5]

(d) Find the $y$-coordinates of the points of intersection of the graphs of $f$ and $g$.

[5]

**2.** A quadratic function $f$ is given by $f(x) = (x-h)^2 + k$.

The vertex of the graph of $f$ is at $(1, -6)$, and the graph crosses the $y$-axis at the point $(0, c)$.

(a) Write down the value of $h$ and of $k$.

[2]

(b) Find the value of $c$.

[2]

Let $g(x)=(x-3)^2-18$. The graph of $g$ is obtained by a reflection of the graph of $f$ in the $y$-axis, followed by a translation of $\begin{pmatrix} p \\ q \end{pmatrix}$.

(c) Find the value of $p$ and of $q$.

[5]

(d) Find the $y$-coordinate of the point of intersection of the graphs of $f$ and $g$.

[5]

**3.** A quadratic function $f$ is given by $f(x)=-(x-h)^2+k$.

The $x$-coordinate of the vertex of the graph of $f$ is 1, and the graph crosses the $y$-axis at the point $(0, 3)$.

(a) Write down the value of $h$.

[1]

(b) Find the value of $k$.

[2]

Let $g(x)=-3x^2+3$. The graph of $g$ is obtained from $f$ by a translation of $\begin{pmatrix} p \\ q \end{pmatrix}$, followed by a vertical stretch of scale factor $r$.

(c) Find the value of $p$, of $q$ and of $r$.

[6]

(d) Find the coordinates of the points of intersection of the graphs of $f$ and $g$.

[8]

**4.** A quadratic function $f$ is given by $f(x)=ax^2+bx+c$.

The vertex of the graph of $f$ is at $(-2, 2)$, and the graph crosses the $y$-axis at the point $(0, 6)$.

(a) Find the value of $a$, of $b$ and of $c$.

[4]

Let $g(x)=5x^2-2$. The graph of $g$ is obtained from $f$ by a vertical stretch of scale factor $r$, followed by a translation of $\begin{pmatrix} p \\ q \end{pmatrix}$.

(b) Find the value of $p$, of $q$ and of $r$.

[6]

(c) Find the coordinates of the points of intersection of the graphs of $f$ and $g$.

[8]

# 11 Paper 2 Section A – Solve the composite equation



**Example**

Let $f(x) = x^2 + 3x - 4$ and $g(x) = x + 6$, for $x \in \mathbb{R}$.

(a) Find $f(10)$.

[2]

(b) Find $(g \circ f)(x)$.

[2]

(c) Solve $(g \circ f)(x) = 0$.

[3]

**Solution**

(a) $f(10)$

$= (10)^2 + 3(10) - 4$ (M1) for substitution

$= 126$ A1 N2

[2]

(b) $(g \circ f)(x)$

$= g(f(x))$

$= f(x) + 6$ (M1) for composite function

$= x^2 + 3x - 4 + 6$

$= x^2 + 3x + 2$ A1 N2

[2]

(c) $(g \circ f)(x) = 0$

$x^2 + 3x + 2 = 0$

$(x+2)(x+1) = 0$ (M1) for factorization

$x = -2$ or $x = -1$ A2 N3

[3]

## Exercise 11

**1.** Let $f(x)=2x^2+8x-7$ and $g(x)=x-17$, for $x\in\mathbb{R}$.

(a) Find $f(6)$.

[2]

(b) Find $(g\circ f)(x)$.

[2]

(c) Solve $(g\circ f)(x)=0$.

[3]

**2.** Let $f(x)=x^2+2x-5$ and $g(x)=x+1$, for $x\in\mathbb{R}$.

(a) Find $f(-2)$.

[2]

(b) Find $(f\circ g)(x)$.

[2]

(c) Solve $(f\circ g)(x)=0$.

[3]

**3.** Let $f(x)=x^3$ and $g(x)=3x-4$, for $x\in\mathbb{R}$.

(a) Find $(g\circ f)(x)$.

[2]

(b) Find $(g\circ f)(3)$.

[2]

(c) Solve $(g\circ f)(x)=1025$.

[2]

**4.** Let $f(x)=5x+1$ and $g(x)=x^4$, for $x\in\mathbb{R}$.

(a) Find $(f\circ g)(x)$.

[2]

(b) Find $(f\circ g)(-3)$.

[2]

(c) Solve $(f\circ g)(x)=1281$.

[3]

# Chapter

# 4

# Exponential and Logarithmic Functions



**SUMMARY POINTs**

- ✓ $y = a^x$ : Exponential function, where $a \neq 1$
- ✓ Methods of solving an exponential equation $a^x = b$:
  1. Change $b$ into $a^y$ such that $a^x = a^y \Rightarrow x = y$
  2. Take logarithm for both sides
- ✓ $y = \log_a x$ : Logarithmic function, where $a > 0$
- ✓ $y = \log x = \log_{10} x$ : Common Logarithmic function
- ✓ $y = \ln x = \log_e x$ : Natural Logarithmic function, where $e = 2.71828...$ is an exponential number

## SUMMARY POINTs

✓ Laws of logarithm:

1. $x = a^y \Leftrightarrow y = \log_a x$
2. $\log_a 1 = 0$
3. $\log_a a = 1$
4. $\log_a p + \log_a q = \log_a pq$
5. $\log_a p - \log_a q = \log_a \frac{p}{q}$
6. $\log_a p^n = n \log_a p$
7. $\log_b a = \frac{\log_c a}{\log_c b}$

where $a$, $b$, $c$, $p$, $q$ and $x > 0$

✓ Properties of the graphs of $y = a^x$:

| $a > 1$ | $0 < a < 1$ |
|---|---|
| $y$-intercept $= 1$ | |
| $y$ increases as $x$ increases | $y$ decreases as $x$ increases |
| $y$ tends to zero as $x$ tends to negative infinity | $y$ tends to zero as $x$ tends to positive infinity |
| Horizontal asymptote: $y = 0$ | |

✓ Properties of the graphs of $y = \log_a x$:

| $a > 1$ | $0 < a < 1$ |
|---|---|
| $x$-intercept $= 1$ | |
| $y$ increases as $x$ increases | $y$ decreases as $x$ increases |
| $x$ tends to zero as $y$ tends to negative infinity | $x$ tends to zero as $y$ tends to positive infinity |
| Vertical asymptote: $x = 0$ | |

**Solutions of Chapter 4**

# 12 Paper 1 Section A – Using laws of logarithm to perform simplifications

**Example**

Find the value of each of the following, giving your answer as an integer.

(a) $\log_9 729$

[2]

(b) $\log_9 162 - \log_9 2$

[3]

(c) $\log_9 \frac{1}{36} + \log_9 4$

[3]

**Solution**

(a) $\log_9 729$

$= \log_9 9^3$ (A1) for valid approach

$= 3$ A1 N2

[2]

(b) $\log_9 162 - \log_9 2$

$= \log_9 \frac{162}{2}$ (A1) for correct formula

$= \log_9 81$

$= \log_9 9^2$ (A1) for valid approach

$= 2$ A1 N2

[3]

(c) $\log_9 \frac{1}{36} + \log_9 4$

$= \log_9 \frac{1}{9}$ (A1) for correct formula

$= \log_9 9^{-1}$ (A1) for valid approach

$= -1$ A1 N2

[3]

## Exercise 12

**1.** Find the value of each of the following, giving your answer as an integer.

(a) $\log_5 25$

[2]

(b) $\log_5 0.5 + \log_5 10$

[2]

(c) $\log_5 4 - \log_5 500$

[3]

**2.** Find the value of each of the following, giving your answer as an integer.

(a) $\log_{0.5} 2$

[2]

(b) $\log_{0.5} \dfrac{1}{7} + \log_{0.5} 7$

[2]

(c) $\log_{0.5} 24 - \log_{0.5} 3$

[3]

**3.** Find the value of

(a) $\log_2 112 - \log_2 7$;

[3]

(b) $27^{\log_3 2}$.

[4]

**4.** Find the value of

(a) $\log_3 \dfrac{1}{3} + \log_3 45 - \log_3 15$;

[3]

(b) $25^{\log_5 7}$.

[4]

# 13 Paper 1 Section A – Composite and inverse functions



**Example**

Let $f(x) = \log_2 x^2$, for $x > 0$.

(a) Show that $f^{-1}(x) = 2^{\frac{1}{2}x}$. [2]

(b) Write down the range of $f^{-1}$. [1]

Let $g(x) = \log_2 x^3$, for $x > 0$.

(c) Find the value of $(f^{-1} \circ g)(4)$, giving your answer as an integer. [4]

**Solution**

(a)

| | | |
|---|---|---|
| $y = \log_2 x^2$ | | |
| $x = \log_2 y^2$ | (M1) for swapping variables | |
| $2^x = y^2$ | A1 | |
| $\sqrt{2^x} = y$ | | |
| $2^{\frac{1}{2}x} = y$ | | |
| $\therefore f^{-1}(x) = 2^{\frac{1}{2}x}$ | AG | N0 |

[2]

(b) Range of $f^{-1}$: $\{y : y \in \mathbb{R}, y > 0\}$ A1 N1

[1]

(c)

| | | |
|---|---|---|
| $g(4)$ | | |
| $= \log_2 4^3$ | | |
| $= \log_2 64$ | A1 | |
| $(f^{-1} \circ g)(4)$ | | |
| $= f^{-1}(g(4))$ | | |
| $= 2^{\frac{1}{2}(\log_2 64)}$ | (M1) for substitution | |
| $= 2^{\log_2 8}$ | (M1) for correct formula | |
| $= 8$ | A1 | N1 |

[4]

## Exercise 13

**1.** Let $f(x)=\log_5 \sqrt[3]{x}$, for $x>0$.

(a) Show that $f^{-1}(x)=5^{3x}$.

[2]

(b) Write down the range of $f^{-1}$.

[1]

Let $g(x)=\log_5 x^2$, for $x>0$.

(c) Find the value of $(f^{-1}\circ g)(5)$, giving your answer as an integer.

[4]

**2.** Let $f(x)=e^{4x}$.

(a) Show that $f^{-1}(x)=0.25\ln x$.

[2]

(b) Write down the domain of $f^{-1}$.

[1]

Let $g(x)=(e^x-1)^3$.

(c) Find the value of $(g\circ f^{-1})(16)$, giving your answer as an integer.

[4]

**3.** Let $f(x)=\ln x+3$, for $x>0$.

(a) Show that $f^{-1}(x)=e^{x-3}$.

[2]

(b) Write down the range of $f^{-1}$.

[1]

Let $g(x)=e^{(x+1)(x-3)}$.

(c) Find the value of $(f\circ g)(2)$, giving your answer as an integer.

[4]

**4.** Let $f(x)=2^{3x}$.

(a) Show that $f^{-1}(x)=\frac{1}{3}\log_2 x$.

[2]

(b) Write down the range of $f^{-1}$.

[1]

Let $g(x)=(1+\log_2 x)^2$.

(c) Express $(g\circ f)(x)$ in the form $ax^2+bx+c$, where $a$, $b$ and $c$ are integers.

[4]

# 14 Paper 1 Section A – Solving exponential and logarithmic equations



## Example

Solve $\log_3 x + \log_3 (x+8) = 2$, for $x > -8$.

[7]

## Solution

$\log_3 x + \log_3 (x+8) = 2$

$\log_3 x(x+8) = 2$ (A1) for correct formula

$\log_3 (x^2 + 8x) = 2$

$x^2 + 8x = 3^2$ (A1) for valid approach

$x^2 + 8x - 9 = 0$ A1

$(x+9)(x-1) = 0$ (M1) for factorization

$x = -9$ (*Rejected*) or $x = 1$ A1

$\therefore x = 1$ A2 N3

[7]

## Exercise 14

**1.** Solve $\log_2 16x - \log_2 (2-x) = 4$, for $0 < x < 2$.

[5]

**2.** Solve $2^{x^2} \cdot 2^{2(3x+4)} = 8$.

[7]

**3.** Consider $f(x) = \log_k \left(\frac{8x - x^2}{4}\right)$, for $0 < x < 8$, where $k > 0$. The equation $f(x) = 2$ has exactly one solution. Find the value of $k$.

[7]

**4.** Consider $f(x) = \log_3 (6x - kx^2)$, for $0 < x < \frac{6}{k}$, where $k > 0$. The equation $f(x) = 1$ has two distinct real solutions. Find the range of values of $k$.

[7]

# 15 Paper 2 Section A – Curve sketching

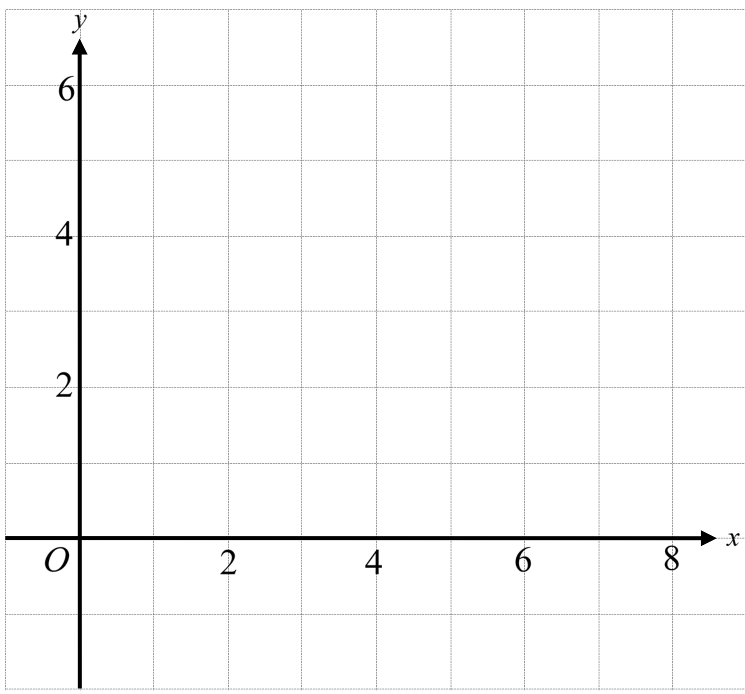
**Example**

Let $f(x)=\dfrac{5-3x^2}{e^x}$, for $0 \le x \le 8$.

(a) Find the $x$-intercept of the graph of $f$. [2]

(b) The graph of $f$ has a minimum at the point A. Write down the coordinates of A. [2]

(c) On the following grid, sketch the graph of $f$. [3]



## Solution

(a) $f(x)=0$ (M1) for equation equals to 0

$\frac{5-3x^2}{e^x}=0$

$x=1.29$ A1 N2

[2]

(b) The minimum point is $(2.63, -1.14)$. A2 N2

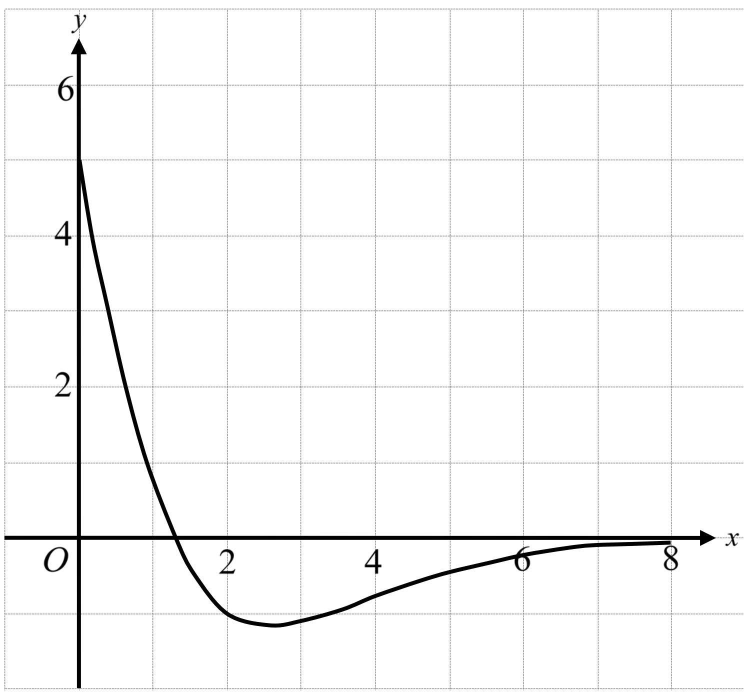
[2]

(c) For correct domain and endpoints at $x=0$ and $x=8$ A1

For correct maximum point A1

For correct concavity A1 N3

[3]



## Exercise 15

**1.** Let $f(x)=\dfrac{7x^2-2}{e^x}$, for $0\le x\le 8$.

(a) Find the $x$-intercept of the graph of $f$.

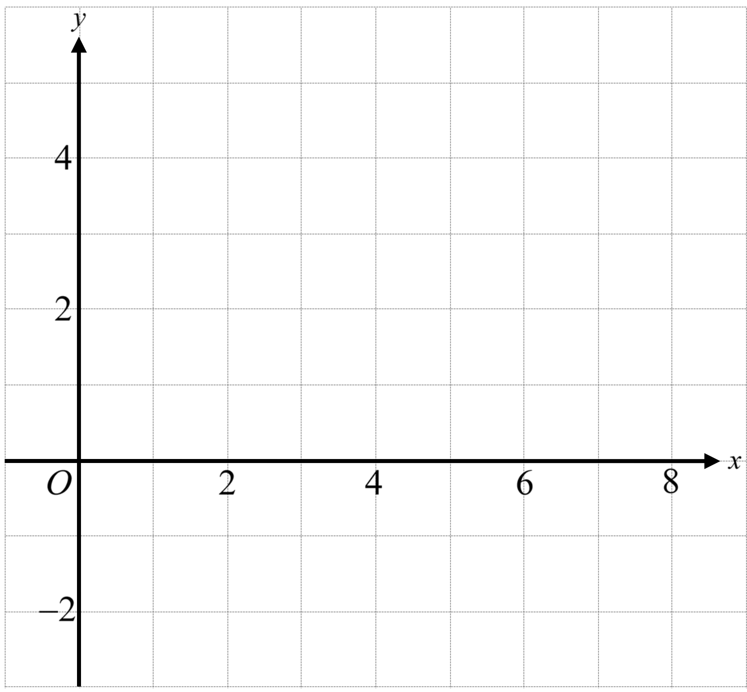
[2]

(b) The graph of $f$ has a maximum at the point A. Write down the coordinates of A.

[2]

(c) On the following grid, sketch the graph of $f$.

[3]



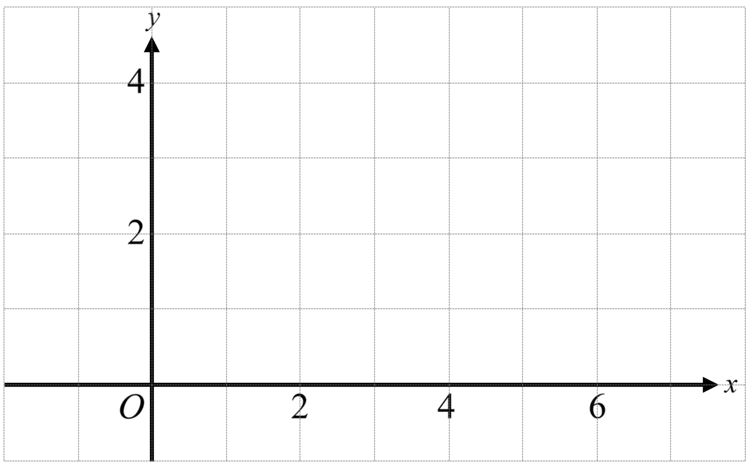
**2.** Let $f(x)=\dfrac{x^3+2x+3}{e^x}$, for $-1\le x\le 7$.

(a) Find the $x$-intercept and the $y$-intercept of the graph of $f$. [3]

(b) The graph of $f$ has a maximum at the point A. Write down the coordinates of A. [2]

(c) On the following grid, sketch the graph of $f$. [3]



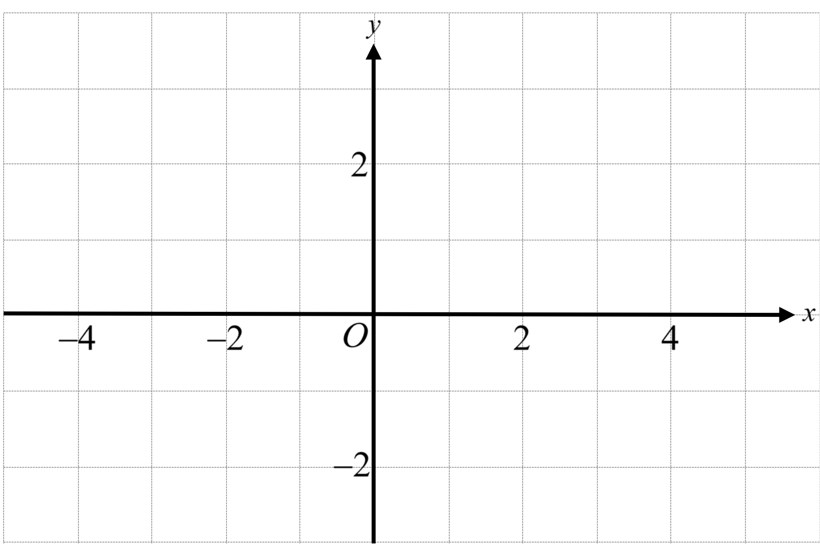



**3.** Let $f(x)=e^{0.3x}-3$, for $-5\le x\le 5$.

(a) Find the $x$-intercept of the graph of $f$. [2]

(b) Write down the equation of the horizontal asymptote of $f$. [2]

(c) On the following grid, sketch the graph of $f$. [3]



**4.** Let $f(x)=\dfrac{e^x}{4x-6}$, for $-3\le x\le 4$.

(a) The graph of $f$ has a minimum at the point A. Write down the coordinates of A.

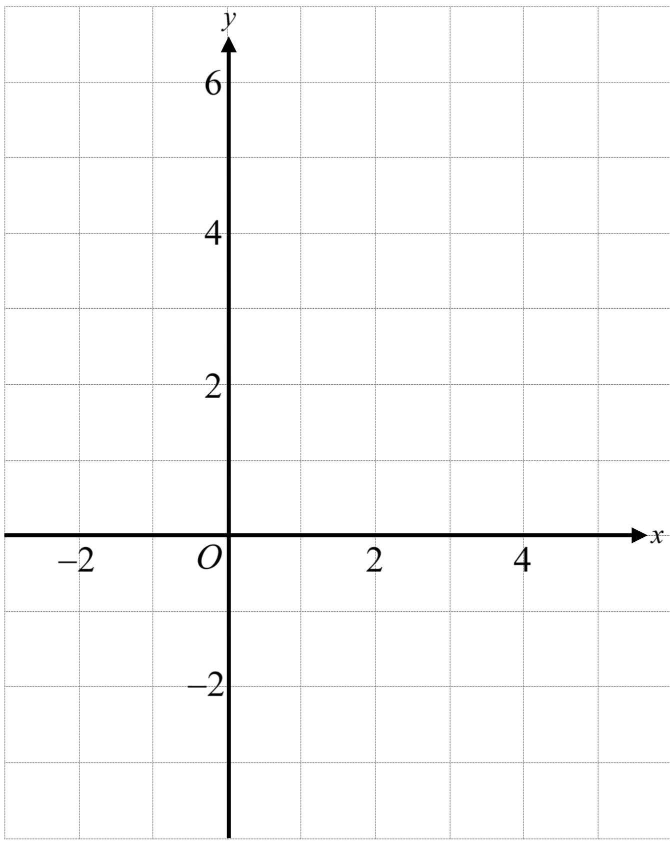
[2]

(b) Write down the equation of the vertical asymptote of $f$.

[2]

(c) On the following grid, sketch the graph of $f$.

[3]



# 16 Paper 2 Section B – Comparing two models



**Example**

The number of insects in two colonies, A and B, starts increasing at the same time. The number of insects in colony A after $t$ months is modeled by the function $A(t) = 240e^{0.3t}$.

(a) Find the initial number of insects in colony A.

[2]

(b) Find the number of insects in colony A after seven months.

[3]

(c) How long does it take for the number of insects in colony A to reach 800?

[3]

The number of insects in colony B after $t$ months is modeled by the function $B(t) = 360e^{kt}$.

(d) After ten months, there are 1000 insects in colony B. Find the value of $k$.

[3]

(e) The number of insects in colony A first exceeds the number of insects in colony B after $n$ months, where $n \in \mathbb{Z}$. Find the value of $n$.

[4]

**Solution**

(a) Initial number of insects

$= 240e^{0.3(0)}$ (A1) for substitution

$= 240$ A1 N2

[2]

(b) Number of insects in colony A after seven months

$= 240e^{0.3(7)}$ (A1) for substitution

$= 1959.880779$ (A1) for correct working

$= 1960$ A1 N3

[3]

(c) $A(t) = 800$ (A1) for setting equation

$240e^{0.3t} = 800$

$240e^{0.3t} - 800 = 0$

By considering the graph of $y = 240e^{0.3t} - 800$ (M1) for correct working

$t = 4.0132427$

$\therefore$ It takes 4.01 months. A1 N3

[3]

(d) $B(10)=1000$ (M1) for substitution

$360e^{10k}=1000$

$360e^{10k}-1000=0$

By considering the graph of $y=360e^{10k}-1000$ (M1) for correct working

$k=0.1021651$

$\therefore k=0.102$ A1 N3

[3]

(e) $A(t)>B(t)$ (M1) for setting inequality

$A(t)-B(t)>0$

$240e^{0.3t}-360e^{0.1021651t}>0$

By considering the graph of

$y=240e^{0.3t}-360e^{0.1021651t}$ (M1) for correct working

$t>2.0495125$ (A1) for correct working

$\therefore n=3$ A1 N3

[4]

## Exercise 16

**1.** The number of leopards and tigers in a forest start increasing at the same time.

The number of leopards in the forest after $t$ years is modeled by the function $A(t)=2500e^{0.075t}$.

(a) Find the initial number of leopards.

[2]

(b) Find the number of leopards after ten years.

[3]

(c) How long does it take for the number of leopards to reach 8000?

[3]

The number of tigers in the forest after $t$ years is modeled by the function $B(t)=ke^{\frac{180}{k}t}$, where $k<2000$.

(d) After ten years, there are 5000 tigers. Find the value of $k$.

[3]

(e) The number of tigers first exceeds the number of leopards after $n$ years, where $n\in\mathbb{Z}$. Find the value of $n$.

[4]

**2.** The number of trams and the number of people using trams in a city is studied.

The number of trams in the city after $t$ years is modeled by the function $A(t) = 420 \times 1.15^t$.

(a) Find the initial number of trams.

[2]

(b) Find the number of trams after six years.

[2]

(c) How long does it take for the number of trams to reach 750?

[3]

The number of people using trams in the city after $t$ years is modeled by the function

$$B(t) = \frac{4680000}{70e^{-kt} + 130}.$$

(d) After five years, there are 27500 people using trams. Find the value of $k$.

[3]

(e) The number of trams first exceeds five times the number of people using trams after $n$ years, where $n \in \mathbb{Z}$. Find the value of $n$.

[4]

**3.** The number of food delivery cars and the number of people using food delivery cars in a town is studied.

The number of food delivery cars in the town after $t$ weeks is modeled by the function $A(t) = 1050 \times 1.25^t$.

(a) Find the initial number of food delivery cars.

[2]

(b) Find the number of food delivery cars after sixteen weeks.

[2]

(c) How long does it take for the number of food delivery cars to reach 4200?

[3]

The number of people using food delivery cars in the town after $t$ weeks is modeled by the function $B(t) = \dfrac{410000}{75k + 95e^{-kt}}$.

(d) After twelve weeks, there are 4600 people using food delivery cars. Find the value of $k$.

[3]

(e) The number of food delivery cars first exceeds double the number of people using food delivery cars after $n$ weeks, where $n \in \mathbb{Z}$. Find the value of $n$.

[4]

**4.** The air pressure in two machines, A and B, are recorded in an experiment.

The air pressure in machine A after $t$ minutes is modeled by the function $P(t)=4e^{0.12t}$.

(a) Find the air pressure in machine A after half an hour.

[2]

(b) How long does it take for the air pressure in machine A to reach 8 units?

[3]

The air pressure in machine B after $t$ minutes is modeled by the function $Q(t)=Q_0e^{kt}$. It is recorded that the initial air pressure and the air pressure in machine B after half an hour are 3.5 units and 171 units respectively.

(c) Find the value of $Q_0$ and of $k$.

[3]

(d) The sum of air pressures in two machines first exceeds 400 units after $n$ minutes, where $n \in \mathbb{Z}$. Find the value of $n$.

[4]

# Chapter

# 9

# Equations of Straight Lines

9

**SUMMARY POINTs**

✓ Consider the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on a $x$ - $y$ plane:

1. $m = \dfrac{y_2 - y_1}{x_2 - x_1}$: Slope of $PQ$
2. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ : Distance between $P$ and $Q$
3. $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$: The mid-point of $PQ$

✓ Forms of straight lines with slope $m$ and $y$-intercept $c$:

1. $y = mx + c$: Slope-intercept form
2. $Ax + By + C = 0$: General form

✓ Ways to find the $x$-intercept and the $y$-intercept of a line:

1. Substitute $y = 0$ and make $x$ the subject to find the $x$-intercept
2. Substitute $x = 0$ and make $y$ the subject to find the $y$-intercept

**Solutions of Chapter 9**

# 30 Paper 1 Section A – Finding the equation of a straight line

## Example

A straight line $L$ passes through the points $(3, 2)$ and $(10, 16)$.

(a) Find the equation of $L$, giving the answer in general form. [3]

(b) Write down the $x$-intercept and the $y$-intercept of $L$. [2]

## Solution

(a) The gradient of $L$

$=\frac{16-2}{10-3}$ (M1) for valid approach

$=2$

The equation of $L$:

$y-2=2(x-3)$ A1

$y-2=2x-6$

$2x-y-4=0$ A1 N2

[3]

(b) The $x$-intercept of $L$ is 2 A1

The $y$-intercept of $L$ is $-4$ A1 N2

[2]

## Exercise 30

**1.** A straight line $L$ passes through the points $(10, 6)$ and $(20, 11)$.

(a) Find the equation of $L$, giving the answer in general form. [3]

(b) Write down the $x$-intercept and the $y$-intercept of $L$. [2]

**2.** A straight line $L$ passes through the points $(-4, -8)$ and $(2, -26)$.

(a) Find the equation of $L$, giving the answer in general form. [3]

(b) Write down the $x$-intercept and the $y$-intercept of $L$. [2]

**3.** A straight line $L_1$ passes through the points $(5, 1)$ and $(17, 37)$.

(a) Find the equation of $L_1$, giving the answer in general form.

[3]

(b) The equation of another straight line, $L_2$, is given as $3x + y - 100 = 0$. Are $L_1$ and $L_2$ parallel? Explain your answer.

[2]

**4.** A straight line $L_1$ passes through the points $(-4, 0)$ and $(4, 40)$.

(a) Find the equation of $L_1$, giving the answer in general form.

[3]

(b) The equation of another straight line, $L_2$, is given as $x + 5y + 150 = 0$. Are $L_1$ and $L_2$ perpendicular? Explain your answer.

[2]

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# 31 Paper 1 Section A – Finding the equations of parallel and perpendicular lines

## Example

The equation of a straight line $L_1$ is given as $2x+y-10=0$.

(a) Write down the gradient and the $x$-intercept of $L_1$.

[2]

(b) Find the equation of another straight line $L_2$ such that $L_2$ is parallel to $L_1$ and $L_2$ passes through $(4, 8)$, giving the answer in general form.

[3]

## Solution

| | | | |
|---|---|---|---|
| (a) | The gradient of $L_1$ is $-2$ | A1 | |
| | The $x$-intercept of $L_1$ is 5 | A1 | N2 |

[2]

| | | | |
|---|---|---|---|
| (b) | The gradient of $L_2$ is $-2$ | (A1) for correct gradient | |
| | The equation of $L_2$: | | |
| | $y-8=-2(x-4)$ | A1 | |
| | $y-8=-2x+8$ | | |
| | $2x+y-16=0$ | A1 | N2 |

[3]

## Exercise 31

**1.** The equation of a straight line $L_1$ is given as $x-2y+16=0$.

(a) Write down the gradient and the $y$-intercept of $L_1$.

[2]

(b) Find the equation of another straight line $L_2$ such that $L_2$ is parallel to $L_1$ and $L_2$ passes through $(-2, 5)$, giving the answer in general form.

[3]

**2.** The equation of a straight line $L_1$ is given as $3x+2y-4=0$.

(a) Write down the gradient and the $x$-intercept of $L_1$.

[2]

(b) Find the equation of another straight line $L_2$ such that $L_2$ is parallel to $L_1$ and $L_2$ passes through $(1,-7)$, giving the answer in general form.

[3]

**3.** The equation of a straight line $L_1$ is given as $3x+y+21=0$.

(a) Write down the gradient and the $x$-intercept of $L_1$.

[2]

(b) Find the equation of another straight line $L_2$ such that $L_2$ is perpendicular to $L_1$ and they intersect at the $x$-axis, giving the answer in general form.

[3]

**4.** The equation of a straight line $L_1$ is given as $2x-4y-17=0$.

(a) Write down the gradient and the $y$-intercept of $L_1$.

[2]

(b) Find the equation of another straight line $L_2$ such that $L_2$ is perpendicular to $L_1$ and they intersect at the $y$-axis, giving the answer in general form.

[3]