AA HL Practice Set 1 Paper 1 Solution

Section A

1. (a) The mean

$= \frac{300}{15}$ (M1) for valid approach

$= 20$ A1

[2]

(b) (i) -40 A1

(ii) The new variance

$= (-2)^2(9)$ (M1) for valid approach

$= 36$ A1

(iii) 6 A1

[4]

2. (a) The gradient of L_1

$= \frac{32-0}{24-8}$ (M1) for valid approach

$= 2$

The equation of L_1:

$y - 0 = 2(x-8)$ A1

$y = 2x - 16$

$2x - y - 16 = 0$ A1

[3]

(b) $2 \times -\frac{1}{-a} = -1$ (M1) for valid approach

$2 = -a$

$a = -2$ A1

[2]

3. (a) L.H.S.

$=(2n+1)^2+(2n+3)^2+(2n+5)^2$

$=4n^2+4n+1+4n^2+12n+9+4n^2+20n+25$ M1A1

$=12n^2+36n+35$

$=12n^2+36n+33+2$ M1

$=3(4n^2+12n+11)+2$

$=$R.H.S. AG

[3]

(b) $2n+1$, $2n+3$ and $2n+5$ are three consecutive odd numbers. R1

$(2n+1)^2+(2n+3)^2+(2n+5)^2$

$=3(4n^2+12n+11)+2$ A1

Also $3(4n^2+12n+11)$ is a multiple of 3. R1

Thus, the sum of the squares of any three consecutive odd numbers is greater than a multiple of 3 by 2. AG

[3]

4. $f(x)=px^3+qx^2-2x+1$

$f'(x)=p(3x^2)+q(2x)-2(1)+0$ (A1) for correct derivatives

$f'(x)=3px^2+2qx-2$

$f'(1)=-1\div-\frac{1}{15}$

$\therefore 3p(1)^2+2q(1)-2=15$ (M1) for setting equation

$3p+2q=17$

$2q=17-3p$ A1

$f^{-1}(41)=2$

$\therefore f(2)=41$ (M1) for valid approach

$p(2)^3+q(2)^2-2(2)+1=41$ A1

$8p+4q-3=41$

$\therefore 8p+2(17-3p)-3=41$ (M1) for substitution

$8p+34-6p-3=41$

$2p=10$

$p=5$ A1

$\therefore q=\frac{17-3(5)}{2}$

$q=1$ A1

[8]

5. (a) $a=\dfrac{37-(-5)}{2}$ M1

$a=21$ A1

$b=\dfrac{2\pi}{2(11-2)}$ M1

$b=\dfrac{\pi}{9}$ A1

$d=\dfrac{37+(-5)}{2}$ M1

$d=16$

$\therefore\ f(t)=21\sin\dfrac{\pi}{9}(t+2.5)+16$ AG

[5]

(b) The coordinates of P'

$=(3(2)+17,\ 37+8)$ A1

$=(23,\ 45)$ A1

[2]

6. (a) $g(x)$

$=3f(x-1)$

$=3(4(x-1)^4+3(x-1)^2-1)$ (A1) for substitution

$=3(4(x^4-4x^3+6x^2-4x+1)+3(x^2-2x+1)-1)$ M1A1

$=3(4x^4-16x^3+24x^2-16x+4+3x^2-6x+3-1)$ M1

$=3(4x^4-16x^3+27x^2-22x+6)$

$=12x^4-48x^3+81x^2-66x+18$ A1

[5]

(b) The sum of the roots

$=-\dfrac{-48}{12}$ M1

$=4$ A1

[2]

7. $1+f(|x|)\le |x|$

$1+\dfrac{2|x|^3-5|x|^2-37}{|x|+37}\le |x|$ M1

$\dfrac{2|x|^3-5|x|^2-37}{|x|+37}\le |x|-1$

$2|x|^3-5|x|^2-37\le (|x|-1)(|x|+37)$

$2|x|^3-5|x|^2-37\le |x|^2+36|x|-37$

$2|x|^3-6|x|^2-36|x|\le 0$ (A1) for correct inequality

$2|x|\left(|x|^2-3|x|-18\right)\le 0$

$|x|^2-3|x|-18\le 0$ M1

$(|x|+3)(|x|-6)\le 0$

$\therefore 0\le |x|\le 6$ A1

$\therefore 1<x\le 6$ A1

[5]

8. When $n=2$,

$$\text{L.H.S.} = \binom{2}{2}$$

$$\text{L.H.S.} = 1$$

$$\text{R.H.S.} = \frac{2(2+1)(2-1)}{6}$$

$$\text{R.H.S.} = 1$$

Thus, the statement is true when $n=2$. R1

Assume that the statement is true when $n=k$. M1

$$\binom{2}{2}+\binom{3}{2}+\ldots+\binom{k}{2} = \frac{k(k+1)(k-1)}{6}$$

When $n=k+1$,

$$\binom{2}{2}+\binom{3}{2}+\ldots+\binom{k}{2}+\binom{k+1}{2}$$

$$= \frac{k(k+1)(k-1)}{6} + \binom{k+1}{2}$$ M1A1

$$= \frac{k(k+1)(k-1)}{6} + \frac{(k+1)(k)}{2}$$ A1

$$= \frac{k(k+1)(k-1)}{6} + \frac{3k(k+1)}{6}$$

$$= \frac{k(k+1)}{6}(k-1+3)$$

$$= \frac{k(k+1)(k+2)}{6}$$

$$= \frac{(k+1)((k+1)+1)((k+1)-1)}{6}$$ A1

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n \in \mathbb{Z}^+$, $n \geq 2$. R1

[7]

9. (a) 1 A1

[1]

(b) $\int_1^a \frac{1}{e^2-1}e^{3-x}\mathrm{d}x = \frac{1}{2}$ A1

$\left[-\frac{1}{e^2-1}e^{3-x}\right]_1^a = \frac{1}{2}$ A1

$-\frac{1}{e^2-1}e^{3-a} - \left(-\frac{1}{e^2-1}e^2\right) = \frac{1}{2}$

$\frac{-e^{3-a}+e^2}{e^2-1} = \frac{1}{2}$ M1

$-e^{3-a}+e^2 = \frac{1}{2}e^2 - \frac{1}{2}$

$e^{3-a} = \frac{e^2+1}{2}$ A1

$3-a = \ln\left(\frac{e^2+1}{2}\right)$

$a = 3-\ln\left(\frac{e^2+1}{2}\right)$

Thus, the median is $3-\ln\left(\frac{e^2+1}{2}\right)$. AG

[4]

Section B

10. (a) (i) $\{y: 0 \le y \le 1, y \in \mathbb{R}\}$ A2

(ii) $f(x) = 1$

$\therefore \cos^4 x = 1$

$\cos^2 x = -1$ (*Rejected*) or $\cos^2 x = 1$ (M1) for valid approach

$\cos x = -1$ or $\cos x = 1$

$x = \pi$ or $x = 0$, $x = 2\pi$ (A1) for correct values

Thus, there are 3 solutions. A1

[5]

(b) $f'(x) = (4\cos^3 x)(-\sin x)$ (A1) for chain rule

$f'(x) = -4\sin x \cos^3 x$ A1

[2]

(c) The total area of the regions

$= \int_0^{\pi} (\cos^4 x)(2\sin x)\mathrm{d}x$ (A1) for definite integral

Let $u = \cos x$
$\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x \Rightarrow (-1)\mathrm{d}u = \sin x\mathrm{d}x$
$x = \pi \Rightarrow u = \cos\pi = -1$
$x = 0 \Rightarrow u = \cos 0 = 1$

(A1) for substitution

$= \int_1^{-1} -2u^4 \mathrm{d}u$ M1A1

$= \left[-\frac{2}{5}u^5\right]_1^{-1}$ A1

$= -\frac{2}{5}(-1)^5 - \left(-\frac{2}{5}(1)^5\right)$ (M1) for substitution

$= \frac{4}{5}$ A1

[7]

11. (a) $\frac{dy}{dx}=h(x)\cdot(y+1)$

$\frac{1}{y+1}dy=\sin x dx$ (M1) for valid approach

$\int\frac{1}{y+1}dy=\int\sin x dx$ (A1) for correct approach

$\ln|y+1|=-\cos x+C$ A1

$y+1=e^{-\cos x+C}$ (M1) for valid approach

$y=e^{-\cos x+C}-1$ A1

$0=e^{-\cos 0+C}-1$ (M1) for substitution

$1=e^{-1+C}$

$-1+C=0$

$C=1$ (A1) for correct value

$\therefore y=e^{1-\cos x}-1$ A1

[8]

(b) $\frac{dy}{dx}=h(x)\sqrt{1-(h(x))^2}\cdot(y+1)$

$\frac{dy}{dx}=\sin x\sqrt{1-\sin^2 x}\cdot(y+1)$

$\frac{dy}{dx}=\sin x\cos x\cdot(y+1)$ A1

$\frac{dy}{dx}=\frac{\sin 2x\cdot(y+1)}{2}$

$\frac{dy}{dx}-\left(\frac{1}{2}\sin 2x\right)y=\frac{1}{2}\sin 2x$ A1

The integrating factor

$=e^{\int-\frac{1}{2}\sin 2x dx}$ M1

$=e^{\frac{1}{4}\cos 2x}$ A1

$\therefore e^{\frac{1}{4}\cos 2x}\frac{dy}{dx}-e^{\frac{1}{4}\cos 2x}\left(\frac{1}{2}\sin 2x\right)y=e^{\frac{1}{4}\cos 2x}\left(\frac{1}{2}\sin 2x\right)$ M1

$\therefore e^{\frac{1}{4}\cos 2x}\frac{dy}{dx}-\frac{1}{2}ye^{\frac{1}{4}\cos 2x}\sin 2x=\frac{1}{2}e^{\frac{1}{4}\cos 2x}\sin 2x$

$\frac{d}{dx}\left(ye^{\frac{1}{4}\cos 2x}\right)=\frac{1}{2}e^{\frac{1}{4}\cos 2x}\sin 2x$ A1

$ye^{\frac{1}{4}\cos 2x}=\int\frac{1}{2}e^{\frac{1}{4}\cos 2x}\sin 2x dx$

Let $u=\frac{1}{4}\cos 2x$. M1

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{4}(-\sin 2x)(2)\Rightarrow(-1)\mathrm{d}u=\frac{1}{2}\sin 2x\mathrm{d}x$$

$\therefore ye^{\frac{1}{4}\cos 2x}=\int -e^{u}\mathrm{d}u$ A1

$ye^{\frac{1}{4}\cos 2x}=-e^{u}+C$

$ye^{\frac{1}{4}\cos 2x}=-e^{\frac{1}{4}\cos 2x}+C$

$y=Ce^{-\frac{1}{4}\cos 2x}-1$ A1

$0=Ce^{-\frac{1}{4}\cos 2(0)}-1$ M1

$1=Ce^{-\frac{1}{4}}$

$C=e^{\frac{1}{4}}$ A1

$\therefore y=e^{\frac{1}{4}-\frac{1}{4}\cos 2x}-1$

$y=e^{\frac{1}{4}-\frac{1}{4}(1-2\sin^2 x)}-1$ A1

$y=e^{\frac{1}{2}\sin^2 x}-1$ AG

[12]

12. (a) $z^6 + 1 = 0$

$z^6 = -1$

$z^6 = \cos\pi + i\sin\pi$ A1

$z = \cos\left(\frac{\pi + 2k\pi}{6}\right) + i\sin\left(\frac{\pi + 2k\pi}{6}\right)$ M1

$(k = 0, 1, 2, 3, 4, 5)$

$z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$, $z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$,

$z = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$, $z = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$,

$z = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$ or $z = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$ A2

[4]

(b) $z^6 + 1$

$= z^6 - z^4 + z^2 + z^4 - z^2 + 1$ M1

$= z^2(z^4 - z^2 + 1) + (z^4 - z^2 + 1)$

$= (z^2 + 1)(z^4 - z^2 + 1)$ A1

$z^4 - z^2 + 1 = 0$

$\frac{z^6 + 1}{z^2 + 1} = 0$, where $z^2 \neq -1$ M1

$z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$, $z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ (*Rejected*),

$z = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$, $z = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$,

$z = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$ (*Rejected*) or

$z = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$ A1

[4]

(c) (i) $(z - p)(z - q) = 0$ M1

$z^2 - (p + q)z + pq = 0$

$p + q = \lambda^3 + \lambda + \lambda^{11} + \lambda^9$

$p + q = \lambda^3 + \lambda + \lambda^5(-1) + \lambda^3(-1)$ M1

$p + q = \lambda^3 + \lambda - \lambda^5 - \lambda^3$

$p + q = \lambda - \lambda^5$

$p + q = \lambda - \frac{-1}{\lambda}$ M1

$p+q=\lambda+\dfrac{1}{\lambda}$

$\therefore\ p+q=\sqrt{3}$ A1

$pq=(\lambda^3+\lambda)(\lambda^{11}+\lambda^9)$

$pq=\lambda^{14}+\lambda^{12}+\lambda^{12}+\lambda^{10}$ M1

$pq=\lambda^2(1)+1+1+\lambda^4(-1)$ M1

$pq=\lambda^2-\lambda^4+2$

$pq=\lambda^2-(\lambda^2-1)+2$

$pq=3$ A1

$\therefore\ z^2-\sqrt{3}z+3=0$ A1

(ii) $(z-(2p))(z-(2q))=0$ M1

$z^2-(2p+2q)z+(2p)(2q)=0$

$z^2-2(p+q)z+4pq=0$ A1

$z^2-2\sqrt{3}z+4(3)=0$ M1

$z^2-2\sqrt{3}z+12=0$ A1

[12]