AA HL Practice Set 4 Paper 3 Solution

1. (a) $F(2)$

$= \sum_{r=1}^{2} \sin\frac{\pi}{2(2)} \sin\frac{r\pi}{2}$ (M1) for substitution

$= \sin\frac{\pi}{4} \sum_{r=1}^{2} \sin\frac{r\pi}{2}$

$= \sin\frac{\pi}{4}\left(\sin\frac{\pi}{2} + \sin\pi\right)$ A1

$= \left(\sin\frac{\pi}{4}\right)(1+0)$

$= \sin\frac{\pi}{4}$ A1

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(b) (i) $\cos(x-y) - \cos(x+y)$

$= \cos x\cos y + \sin x\sin y$

$-(\cos x\cos y - \sin x\sin y)$ A2

$= 2\sin x\sin y$ AG

(ii) Let $x = \frac{A+B}{2}$ and $y = \frac{B-A}{2}$.

$\cos(x-y) - \cos(x+y)$

$= \cos\left(\frac{A+B}{2} - \frac{B-A}{2}\right)$

$-\cos\left(\frac{A+B}{2} + \frac{B-A}{2}\right)$ A1

$= \cos\frac{2A}{2} - \cos\frac{2B}{2}$ M1

$= \cos A - \cos B$

$\therefore \cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{B-A}{2}$ AG

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(c) $F(4)$

$$=\sum_{r=1}^{4}\sin\frac{\pi}{2(4)}\sin\frac{r\pi}{4}$$

$$=\sin\frac{\pi}{8}\sum_{r=1}^{4}\sin\frac{r\pi}{4}$$

$$=\sin\frac{\pi}{8}\left(\sin\frac{\pi}{4}+\sin\frac{\pi}{2}+\sin\frac{3\pi}{4}+\sin\pi\right)$$ A1

$$=\sin\frac{\pi}{8}\sin\frac{\pi}{4}+\sin\frac{\pi}{8}\sin\frac{\pi}{2}$$
$$+\sin\frac{\pi}{8}\sin\frac{3\pi}{4}+\sin\frac{\pi}{8}\sin\pi$$

$$=\frac{1}{2}\left(\cos\left(\frac{\pi}{8}-\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{8}+\frac{\pi}{4}\right)\right)$$
$$+\frac{1}{2}\left(\cos\left(\frac{\pi}{8}-\frac{\pi}{2}\right)-\cos\left(\frac{\pi}{8}+\frac{\pi}{2}\right)\right)$$ M1A1
$$+\frac{1}{2}\left(\cos\left(\frac{\pi}{8}-\frac{3\pi}{4}\right)-\cos\left(\frac{\pi}{8}+\frac{3\pi}{4}\right)\right)$$

$$=\frac{1}{2}\left(\begin{array}{l}\cos\left(-\frac{\pi}{8}\right)-\cos\frac{3\pi}{8}+\cos\left(-\frac{3\pi}{8}\right)-\cos\frac{5\pi}{8}\\+\cos\left(-\frac{5\pi}{8}\right)-\cos\frac{7\pi}{8}\end{array}\right)$$

$$=\frac{1}{2}\left(\begin{array}{l}\cos\frac{\pi}{8}-\cos\frac{3\pi}{8}+\cos\frac{3\pi}{8}-\cos\frac{5\pi}{8}\\+\cos\frac{5\pi}{8}-\cos\frac{7\pi}{8}\end{array}\right)$$ A1

$$=\frac{1}{2}\left(\cos\frac{\pi}{8}-\cos\frac{7\pi}{8}\right)$$

$$=\frac{1}{2}\left(2\sin\frac{\frac{\pi}{8}+\frac{7\pi}{8}}{2}\sin\frac{\frac{7\pi}{8}-\frac{\pi}{8}}{2}\right)$$ A1

$$=\sin\frac{\pi}{2}\sin\frac{3\pi}{8}$$

$$=\sin\frac{3\pi}{8}$$ A1

[6]

(d) $F(n)$

$$= \sum_{r=1}^{n} \sin\frac{\pi}{2n}\sin\frac{r\pi}{n}$$

$$= \sum_{r=1}^{n} \frac{1}{2}\left(\cos\left(\frac{\pi}{2n}-\frac{r\pi}{n}\right)-\cos\left(\frac{\pi}{2n}+\frac{r\pi}{n}\right)\right)$$ M1A1

$$= \sum_{r=1}^{n} \frac{1}{2}\left(\cos\left(\frac{\pi}{2n}-\frac{2r\pi}{2n}\right)-\cos\left(\frac{\pi}{2n}+\frac{2r\pi}{2n}\right)\right)$$

$$= \sum_{r=1}^{n} \frac{1}{2}\left(\cos\frac{(1-2r)\pi}{2n}-\cos\frac{(1+2r)\pi}{2n}\right)$$ M1

$$= \frac{1}{2}\left(\begin{array}{l}\cos\frac{(1-2(1))\pi}{2n}-\cos\frac{(1+2(1))\pi}{2n} \\ +\cos\frac{(1-2(2))\pi}{2n}-\cos\frac{(1+2(2))\pi}{2n} \\ +\cdots+\cos\frac{(1-2n)\pi}{2n}-\cos\frac{(1+2n)\pi}{2n}\end{array}\right)$$

$$= \frac{1}{2}\left(\begin{array}{l}\cos\left(-\frac{\pi}{2n}\right)-\cos\frac{3\pi}{2n}+\cos\left(-\frac{3\pi}{2n}\right)-\cos\frac{5\pi}{2n} \\ +\cdots+\cos\frac{(1-2n)\pi}{2n}-\cos\frac{(1+2n)\pi}{2n}\end{array}\right)$$

$$= \frac{1}{2}\left(\begin{array}{l}\cos\frac{\pi}{2n}-\cos\frac{3\pi}{2n}+\cos\frac{3\pi}{2n}-\cos\frac{5\pi}{2n} \\ +\cdots+\cos\frac{(2n-1)\pi}{2n}-\cos\frac{(1+2n)\pi}{2n}\end{array}\right)$$ A1

$$= \frac{1}{2}\left(\cos\frac{\pi}{2n}-\cos\frac{(1+2n)\pi}{2n}\right)$$ M1

$$= \frac{1}{2}\left(2\sin\frac{\frac{\pi}{2n}+\frac{(1+2n)\pi}{2n}}{2}\sin\frac{\frac{(1+2n)\pi}{2n}-\frac{\pi}{2n}}{2}\right)$$ A1

$$= \sin\frac{(2+2n)\pi}{4n}\sin\frac{2n\pi}{4n}$$

$$= \sin\frac{(1+n)\pi}{2n}\sin\frac{\pi}{2}$$

$$\therefore F(n) = \sin\frac{(1+n)\pi}{2n}$$ AG

[6]

(e) $\left|z^r - 1\right|$

$$= \left|\left(\cos\frac{2\pi}{n} + \mathrm{i}\sin\frac{2\pi}{n}\right)^r - 1\right|$$

$$= \left|\cos\frac{2\pi r}{n} + \mathrm{i}\sin\frac{2\pi r}{n} - 1\right|$$ (M1) for valid approach

$$= \sqrt{\left(\cos\frac{2\pi r}{n} - 1\right)^2 + \sin^2\frac{2\pi r}{n}}$$

$$= \sqrt{\cos^2\frac{2\pi r}{n} - 2\cos\frac{2\pi r}{n} + 1 + \sin^2\frac{2\pi r}{n}}$$ M1

$$= \sqrt{2 - 2\cos\frac{2\pi r}{n}}$$

$$= \sqrt{2 - 2\left(1 - 2\sin^2\frac{\pi r}{n}\right)}$$ A1

$$= \sqrt{4\sin^2\frac{\pi r}{n}}$$

$$= 2\sin\frac{\pi r}{n}$$

$$\because -1 \le \sin\frac{\pi r}{n} \le 1$$ R1

$$\therefore \left|z^r - 1\right| \le 2$$ A1

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(f) $G(n)$

$= \sum_{r=1}^{n} \left| z^r - 1 \right|$

$= \sum_{r=1}^{n} 2\sin\frac{\pi r}{n}$

$= \dfrac{2\sum_{r=1}^{n} \sin\frac{\pi}{2n} \sin\frac{\pi r}{n}}{\sin\frac{\pi}{2n}}$ M1

$= \dfrac{2F(n)}{\sin\frac{\pi}{2n}}$ A1

$= \dfrac{2\sin\frac{(1+n)\pi}{2n}}{\sin\frac{\pi}{2n}}$

$= \dfrac{2\cos\left(\frac{\pi}{2} - \frac{(1+n)\pi}{2n}\right)}{\sin\frac{\pi}{2n}}$ A1

$= \dfrac{2\cos\left(\frac{n\pi}{2n} - \frac{\pi + n\pi}{2n}\right)}{\sin\frac{\pi}{2n}}$

$= \dfrac{2\cos\left(-\frac{\pi}{2n}\right)}{\sin\frac{\pi}{2n}}$ M1

$= \dfrac{2\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}$

$= 2\cot\frac{\pi}{2n}$ A1

[5]

2. (a) (i) $I(0)$

$=\int_0^{\pi} x\mathrm{d}x$ M1

$=\left[\frac{1}{2}x^2\right]_0^{\pi}$ A1

$=\frac{1}{2}\pi^2-\frac{1}{2}(0)^2$

$=\frac{1}{2}\pi^2$ AG

(ii) $I(1)$

$=\int_0^{\pi} x\sin x\mathrm{d}x$

Let $\theta=\cos x$. (M1) for valid approach

$\frac{\mathrm{d}\theta}{\mathrm{d}x}=-\sin x \Rightarrow (-1)\frac{\mathrm{d}\theta}{\mathrm{d}x}=\sin x$

$\therefore I(1)$

$=\int_0^{\pi} x(-1)\frac{\mathrm{d}(\cos x)}{\mathrm{d}x}\mathrm{d}x$

$=\left[-x\cos x\right]_0^{\pi}-\int_0^{\pi}\cos x\cdot\frac{\mathrm{d}(-x)}{\mathrm{d}x}\mathrm{d}x$ A1

$=\left[-x\cos x\right]_0^{\pi}-\int_0^{\pi}\cos x\cdot(-1)\mathrm{d}x$ A1

$=\left[-x\cos x\right]_0^{\pi}+\int_0^{\pi}\cos x\mathrm{d}x$

$=\left[-x\cos x\right]_0^{\pi}+\left[\sin x\right]_0^{\pi}$ A1

$=\left[-x\cos x+\sin x\right]_0^{\pi}$

$=(-\pi\cos\pi+\sin\pi)-(0+\sin 0)$

$=\pi$ A1

[7]

(b) (i) $I(n+2)$

$=\int_0^{\pi} x\sin^{n+2} x\mathrm{d}x$

$=\int_0^{\pi} x\sin^{n} x\sin^2 x\mathrm{d}x$ M1

$=\int_0^{\pi} x\sin^{n} x(1-\cos^2 x)\mathrm{d}x$

$=\int_0^{\pi} x\sin^{n} x\mathrm{d}x-\int_0^{\pi} x\sin^{n} x\cos^2 x\mathrm{d}x$ A1

$=I(n)-\int_0^{\pi} x\sin^{n} x\cos^2 x\mathrm{d}x$ AG

(ii) $\int_0^{\pi} x\sin^n x\cos^2 x\mathrm{d}x$

$$=\frac{1}{n+1}\int_0^{\pi} x\cos x\cdot\frac{\mathrm{d}(\sin^{n+1}x)}{\mathrm{d}x}\mathrm{d}x$$

$$=\frac{1}{n+1}\left\{\begin{array}{l}\left[x\cos x\sin^{n+1}x\right]_0^{\pi}\\ -\int_0^{\pi}\sin^{n+1}x\cdot\frac{\mathrm{d}(x\cos x)}{\mathrm{d}x}\mathrm{d}x\end{array}\right\}$$ A1

$$=\frac{1}{n+1}\left\{\begin{array}{l}\left[x\cos x\sin^{n+1}x\right]_0^{\pi}\\ -\int_0^{\pi}\sin^{n+1}x(\cos x-x\sin x)\mathrm{d}x\end{array}\right\}$$ A1

$$=\frac{1}{n+1}\left\{\begin{array}{l}\left[x\cos x\sin^{n+1}x\right]_0^{\pi}\\ -\int_0^{\pi}(\sin^{n+1}x\cos x-x\sin^{n+2}x)\mathrm{d}x\end{array}\right\}$$

$$=\frac{1}{n+1}\left\{\begin{array}{l}\left[x\cos x\sin^{n+1}x\right]_0^{\pi}\\ -\int_0^{\pi}\sin^{n+1}x\cos x\mathrm{d}x+I(n+2)\end{array}\right\}$$ M1

$$=\frac{1}{n+1}\left\{\begin{array}{l}(\pi\cos\pi\sin^{n+1}\pi-0)\\ -\int_0^{\pi}\sin^{n+1}x\cos x\mathrm{d}x+I(n+2)\end{array}\right\}$$ M1

$$=\frac{1}{n+1}\left\{-\int_0^{\pi}\sin^{n+1}x\cos x\mathrm{d}x+I(n+2)\right\}$$

Let $u=\sin x$. (M1) for substitution

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\cos x\Rightarrow \mathrm{d}u=\cos x\mathrm{d}x$$

$x=\pi\Rightarrow u=\sin\pi=0$

$x=0\Rightarrow u=\sin 0=0$

$\therefore \int_0^{\pi} x\sin^n x\cos^2 x\mathrm{d}x$

$$=\frac{1}{n+1}\left\{-\int_0^{0}u^{n+1}\mathrm{d}u+I(n+2)\right\}$$ A1

$$=\frac{1}{n+1}(0+I(n+2))$$

$$=\frac{1}{n+1}I(n+2)$$ A1

(iii) $I(n+2)=I(n)-\frac{1}{n+1}I(n+2)$ A1

$(n+1)I(n+2)=(n+1)I(n)-I(n+2)$

$(n+2)I(n+2)=(n+1)I(n)$ M1

$I(n+2)=\frac{n+1}{n+2}I(n)$ AG

[11]

(c) (i) $I(4)$

$=\frac{2+1}{2+2}I(2)$ M1

$=\frac{3}{4}\left(\frac{0+1}{0+2}I(0)\right)$ M1

$=\frac{3}{4}\left(\frac{1}{2}\cdot\frac{1}{2}\pi^2\right)$

$=\frac{3}{16}\pi^2$ A1

(ii) $I(7)$

$=\frac{5+1}{5+2}I(5)$ M1

$=\frac{6}{7}\left(\frac{3+1}{3+2}I(3)\right)$ M1

$=\frac{6}{7}\left(\frac{4}{5}\right)\left(\frac{1+1}{1+2}I(1)\right)$

$=\frac{6}{7}\left(\frac{4}{5}\right)\left(\frac{2}{3}\pi\right)$

$=\frac{16}{35}\pi$ A1

[6]

(d) $0\le\sin x\le1$ for $0\le x\le\pi$. A1

Therefore, $\sin^2 x\le\sin x\le1$, implies that

$\int_0^{\pi}x\sin^{2n-2}x\cdot\sin^2 x\mathrm{d}x$

$\le\int_0^{\pi}x\sin^{2n-2}x\cdot\sin x\mathrm{d}x\le\int_0^{\pi}x\sin^{2n-2}x\cdot1\mathrm{d}x$. R1

Thus, $I(2n)\le I(2n-1)\le I(2n-2)$ for $n\ge1$. AG

[2]