Chapter 19 Solution

Exercise 79

1. (a) $E(X)=\int_1^3 x\cdot\left(-\frac{1}{2}x+\frac{3}{2}\right)dx$ (A1) for substitution

$E(X)=\int_1^3\left(-\frac{1}{2}x^2+\frac{3}{2}x\right)dx$

$E(X)=\left[-\frac{1}{6}x^3+\frac{3}{4}x^2\right]_1^3$

$E(X)=\left(-\frac{27}{6}+\frac{27}{4}\right)-\left(-\frac{1}{6}+\frac{3}{4}\right)$

$E(X)=\frac{5}{3}$ A1

[2]

(b) $\int_1^a \left(-\frac{1}{2}x+\frac{3}{2}\right)dx=\frac{1}{4}$ A1

$\left[-\frac{1}{4}x^2+\frac{3}{2}x\right]_1^a=\frac{1}{4}$ A1

$\left(-\frac{1}{4}a^2+\frac{3}{2}a\right)-\left(-\frac{1}{4}+\frac{3}{2}\right)=\frac{1}{4}$

$-\frac{1}{4}a^2+\frac{3}{2}a-\frac{5}{4}-\frac{1}{4}=0$

$a^2-6a+6=0$

$a=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(6)}}{2(1)}$

$a=\frac{6+\sqrt{12}}{2}$ (*Rejected*) or $a=\frac{6-\sqrt{12}}{2}$ (A1) for correct value

$\int_1^b \left(-\frac{1}{2}x+\frac{3}{2}\right)dx=\frac{3}{4}$

$-\frac{1}{4}b^2+\frac{3}{2}b-\frac{5}{4}-\frac{3}{4}=0$

$b^2-6b+8=0$

$(b-2)(b-4)=0$

$b=2$ or $b=4$ (*Rejected*) (A1) for correct value

The interquartile range

$=2-\frac{6-\sqrt{12}}{2}$

$=2-(3-\sqrt{3})$

$=-1+\sqrt{3}$ A1

[5]

2. (a) $E(X)=\int_{-1}^{2} x\cdot\frac{1}{3}\,dx$ (A1) for substitution

$E(X)=\left[\frac{1}{6}x^2\right]_{-1}^{2}$

$E(X)=\frac{4}{6}-\frac{1}{6}$

$E(X)=\frac{1}{2}$ A1

[2]

(b) $E(X^2)=\int_{-1}^{2} x^2\cdot\frac{1}{3}\,dx$ (A1) for substitution

$E(X^2)=\left[\frac{1}{9}x^3\right]_{-1}^{2}$

$E(X^2)=\frac{8}{9}-\left(-\frac{1}{9}\right)$

$E(X^2)=1$ A1

[2]

(c) Standard deviation

$=E(X^2)-(E(X))^2$ (A1) for substitution

$=1-\left(\frac{1}{2}\right)^2$

$=\frac{3}{4}$ A1

[2]

3. (a) 2 A1

[1]

(b) $\int_0^a \frac{1}{e^2-1}e^x \mathrm{d}x = \frac{1}{2}$ A1

$\left[\frac{1}{e^2-1}e^x\right]_0^a = \frac{1}{2}$ A1

$\frac{1}{e^2-1}e^a - \frac{1}{e^2-1}e^0 = \frac{1}{2}$

$\frac{e^a-1}{e^2-1} = \frac{1}{2}$ (M1) for valid approach

$e^a - 1 = \frac{1}{2}e^2 - \frac{1}{2}$

$e^a = \frac{e^2+1}{2}$

$a = \ln\left(\frac{e^2+1}{2}\right)$

Thus, the median is $\ln\left(\frac{e^2+1}{2}\right)$. A1

[4]

4. (a) $\sqrt{3}$ A1

[1]

(b) $\int_{-1}^{a}\left|\frac{x}{2}\right|dx=\frac{1}{2}$ A1

$\int_{-1}^{0}-\frac{x}{2}dx+\int_{0}^{a}\frac{x}{2}dx=\frac{1}{2}$

$\left[-\frac{1}{4}x^2\right]_{-1}^{0}+\left[\frac{1}{4}x^2\right]_{0}^{a}=\frac{1}{2}$ A1

$\left(0-\left(-\frac{1}{4}\right)\right)+\left(\frac{1}{4}a^2-0\right)=\frac{1}{2}$

$\frac{1}{4}a^2=\frac{1}{4}$ (M1) for valid approach

$a^2=1$

$a=-1$ (*Rejected*) or $a=1$

Thus, the median is 1. A1

[4]

(c) $\mathrm{E}(X)=\int_{-1}^{\sqrt{3}}x\cdot\left|\frac{x}{2}\right|dx$ (A1) for substitution

$\mathrm{E}(X)=\int_{-1}^{0}-\frac{1}{2}x^2dx+\int_{0}^{\sqrt{3}}\frac{1}{2}x^2dx$

$\mathrm{E}(X)=\left[-\frac{1}{6}x^3\right]_{-1}^{0}+\left[\frac{1}{6}x^3\right]_{0}^{\sqrt{3}}$ A1

$\mathrm{E}(X)=\left(0-\frac{1}{6}\right)+\left(\frac{3\sqrt{3}}{6}-0\right)$

$\mathrm{E}(X)=\frac{3\sqrt{3}-1}{6}$ A1

[3]

Exercise 80

1.	(a)	$\int_{1}^{\frac{3}{2}} k(x-1)\mathrm{d}x + \int_{\frac{3}{2}}^{2} k(x-2)^2\mathrm{d}x = 1$	(M1) for valid approach
		$\left[\frac{k}{2}(x-1)^2\right]_{1}^{\frac{3}{2}} + \left[\frac{k}{3}(x-2)^3\right]_{\frac{3}{2}}^{2} = 1$	A1
		$\left(\frac{k}{8}-0\right)+\left(0-\left(-\frac{k}{24}\right)\right)=1$	
		$\frac{k}{6}=1$	
		$k=6$	A1
			[3]
	(b)	$\mathrm{E}(X)=\int_{1}^{\frac{3}{2}} 6x(x-1)\mathrm{d}x + \int_{\frac{3}{2}}^{2} 6x(x-2)^2\mathrm{d}x$	(A1) for substitution
		$\mathrm{E}(X)=1+\frac{13}{32}$	(A1) for correct values
		$\mathrm{E}(X)=\frac{45}{32}$	A1
			[3]
	(c)	$\int_{1}^{a} 6(x-1)\mathrm{d}x=\frac{1}{2}$	A1
		$\left[3(x-1)^2\right]_{1}^{a}=\frac{1}{2}$	A1
		$3(a-1)^2-0=\frac{1}{2}$	
		$(a-1)^2=\frac{1}{6}$	
		$a-1=\sqrt{\frac{1}{6}}$	
		$a=1.40824829$	
		Thus, the median is 1.41.	A1
			[3]

(d) $\mathrm{P}\left(X > \frac{5}{4}\right) = \int_{\frac{5}{4}}^{\frac{3}{2}} 6(x-1)\,\mathrm{d}x + \int_{\frac{3}{2}}^{2} 6(x-2)^2\,\mathrm{d}x$ (M1) for valid approach

$\mathrm{P}\left(X > \frac{5}{4}\right) = \frac{9}{16} + \frac{1}{4}$ (A1) for correct values

$\mathrm{P}\left(X > \frac{5}{4}\right) = \frac{13}{16}$ A1

[3]

(e) $\mathrm{P}\left(X > \frac{7}{4} \middle| X > \frac{5}{4}\right) = \frac{\mathrm{P}\left(X > \frac{7}{4} \cap X > \frac{5}{4}\right)}{\mathrm{P}\left(X > \frac{5}{4}\right)}$ (M1) for valid approach

$\mathrm{P}\left(X > \frac{7}{4} \middle| X > \frac{5}{4}\right) = \frac{\mathrm{P}\left(X > \frac{7}{4}\right)}{\mathrm{P}\left(X > \frac{5}{4}\right)}$

$\mathrm{P}\left(X > \frac{7}{4} \middle| X > \frac{5}{4}\right) = \frac{\int_{\frac{7}{4}}^{2} 6(x-2)^2\,\mathrm{d}x}{\frac{13}{16}}$ A1

$\mathrm{P}\left(X > \frac{7}{4} \middle| X > \frac{5}{4}\right) = \frac{16}{13}\left(\frac{1}{32}\right)$

$\mathrm{P}\left(X > \frac{7}{4} \middle| X > \frac{5}{4}\right) = \frac{1}{26}$ A1

[3]

2. (a) $\mathrm{E}(X)=\int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}}-\frac{1}{\pi^2}x\left(x+\frac{\pi}{2}\right)\mathrm{d}x+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{4}x\cos x\,\mathrm{d}x$ (A1) for substitution

$\mathrm{E}(X)=-1.832595715+0$ (A1) for correct values

$\mathrm{E}(X)=-1.832595715$

$\mathrm{E}(X)=-1.83$ A1

[3]

(b) The variance of X

$=\mathrm{E}(X^2)-(\mathrm{E}(X))^2$ (M1) for valid approach

$=\int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}}-\frac{1}{\pi^2}x^2\left(x+\frac{\pi}{2}\right)\mathrm{d}x+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{4}x^2\cos x\,\mathrm{d}x$ A1

$-(-1.832595715)^2$

$=6.990969784+0.2337005501-(-1.832595715)^2$

$=3.866263279$

$=3.87$ A1

[3]

(c) $\mathrm{P}(X<0)=\int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}}-\frac{1}{\pi^2}\left(x+\frac{\pi}{2}\right)\mathrm{d}x+\int_{-\frac{\pi}{2}}^{0}\frac{1}{4}\cos x\,\mathrm{d}x$ (M1) for valid approach

$\mathrm{P}(X<0)=\frac{1}{2}+\frac{1}{4}$ (A1) for correct values

$\mathrm{P}(X<0)=\frac{3}{4}$ A1

[3]

(d) $P(X > -\pi \mid X < 0) = \dfrac{P(X > -\pi \cap X < 0)}{P(X < 0)}$ (M1) for valid approach

$P(X > -\pi \mid X < 0) = \dfrac{P(-\pi < X < 0)}{P(X < 0)}$

$P(X > -\pi \mid X < 0)$

$= \dfrac{\int_{-\pi}^{-\frac{\pi}{2}} -\frac{1}{\pi^2}\left(x + \frac{\pi}{2}\right)dx + \int_{-\frac{\pi}{2}}^{0} \frac{1}{4}\cos x dx}{\frac{3}{4}}$ A1

$P(X > -\pi \mid X < 0) = \dfrac{4}{3}\left(\dfrac{1}{8} + \dfrac{1}{4}\right)$

$P(X > -\pi \mid X < 0) = \dfrac{1}{2}$ A1

[3]

(e) $P(X < r) = 0.6$

$P\left(X < -\dfrac{\pi}{2}\right) + P\left(-\dfrac{\pi}{2} < X < r\right) = 0.6$ (M1) for valid approach

$\dfrac{1}{2} + \int_{-\frac{\pi}{2}}^{r} \dfrac{1}{4}\cos x dx = 0.6$

$\left[\dfrac{1}{4}\sin x\right]_{-\frac{\pi}{2}}^{r} = 0.1$

$\dfrac{1}{4}\sin r - \left(-\dfrac{1}{4}\right) = 0.1$ A1

$\sin r = -0.6$

$r = -0.6435011088$

$r = -0.644$ A1

[3]

3. (a) $\int_0^{\pi} k\sin\frac{1}{2}x\mathrm{d}x+\int_{\pi}^{\pi+1} k\mathrm{d}x=1$ M1

$\left[-2k\cos\frac{1}{2}x\right]_0^{\pi}+\left[kx\right]_{\pi}^{\pi+1}=1$ A1

$(0-(-2k))+(k(\pi+1)-k\pi)=1$ A1

$3k=1$

$k=\frac{1}{3}$ AG

[3]

(b) $\mathrm{E}(X)=\int_0^{\pi}\frac{1}{3}x\sin\frac{1}{2}x\mathrm{d}x+\int_{\pi}^{\pi+1}\frac{1}{3}x\mathrm{d}x$ (A1) for substitution

$\mathrm{E}(X)=\frac{4}{3}+1.213864218$ (A1) for correct values

$\mathrm{E}(X)=2.547197551$

$\mathrm{E}(X)=2.55$ A1

[3]

(c) $\int_0^{\pi}\frac{1}{3}\sin\frac{1}{2}x\mathrm{d}x+\int_{\pi}^{a}\frac{1}{3}\mathrm{d}x=\frac{3}{4}$ A1

$\left[-\frac{2}{3}\cos\frac{1}{2}x\right]_0^{\pi}+\left[\frac{1}{3}x\right]_{\pi}^{a}=\frac{3}{4}$ A1

$\left(0-\left(-\frac{2}{3}\right)\right)+\left(\frac{a}{3}-\frac{\pi}{3}\right)=\frac{3}{4}$

$\frac{a}{3}-\frac{\pi}{3}=\frac{1}{12}$

$a-\pi=\frac{1}{4}$

$a=\frac{1}{4}+\pi$

Thus, the upper quartile is $\frac{1}{4}+\pi$. A1

[3]

(d) $P\left(X<\frac{\pi}{2}\right)=\int_0^{\frac{\pi}{2}}\frac{1}{3}\sin\frac{1}{2}x\,dx$ M1

$P\left(X<\frac{\pi}{2}\right)=\left[-\frac{2}{3}\cos\frac{1}{2}x\right]_0^{\frac{\pi}{2}}$

$P\left(X<\frac{\pi}{2}\right)=-\frac{2}{3}\left(\frac{\sqrt{2}}{2}\right)-\left(-\frac{2}{3}\right)$ A1

$P\left(X<\frac{\pi}{2}\right)=\frac{2-\sqrt{2}}{3}$ AG

[2]

(e) $P\left(\frac{\pi}{2}<X<\pi \middle| X>\frac{\pi}{2}\right)=\frac{P\left(\frac{\pi}{2}<X<\pi\cap X>\frac{\pi}{2}\right)}{P\left(X>\frac{\pi}{2}\right)}$ (M1) for valid approach

$P\left(\frac{\pi}{2}<X<\pi \middle| X>\frac{\pi}{2}\right)=\frac{P\left(\frac{\pi}{2}<X<\pi\right)}{P\left(X>\frac{\pi}{2}\right)}$

$P\left(\frac{\pi}{2}<X<\pi \middle| X>\frac{\pi}{2}\right)=\frac{\int_{\frac{\pi}{2}}^{\pi}\frac{1}{3}\sin\frac{1}{2}x\,dx}{1-\frac{2-\sqrt{2}}{3}}$ A1

$P\left(\frac{\pi}{2}<X<\pi \middle| X>\frac{\pi}{2}\right)=\frac{\left[-\frac{2}{3}\cos\frac{1}{2}x\right]_{\frac{\pi}{2}}^{\pi}}{\frac{1+\sqrt{2}}{3}}$

$P\left(\frac{\pi}{2}<X<\pi \middle| X>\frac{\pi}{2}\right)=\frac{0-\left(-\frac{2}{3}\left(\frac{\sqrt{2}}{2}\right)\right)}{\frac{1+\sqrt{2}}{3}}$

$P\left(\frac{\pi}{2}<X<\pi \middle| X>\frac{\pi}{2}\right)=\frac{\frac{\sqrt{2}}{3}}{\frac{1+\sqrt{2}}{3}}$

$P\left(\frac{\pi}{2}<X<\pi \middle| X>\frac{\pi}{2}\right)=\frac{\sqrt{2}}{1+\sqrt{2}}$ A1

[3]

4. (a)

$\mathrm{E}(X)=\int_{-3}^{1} x\cdot\left(\frac{1}{20}x+\frac{3}{20}\right)\mathrm{d}x$

$+\int_{1}^{4} x\cdot\left(-\frac{1}{15}x^2+\frac{4}{15}x\right)\mathrm{d}x$ (A1) for substitution

$\mathrm{E}(X)=\int_{-3}^{1}\left(\frac{1}{20}x^2+\frac{3}{20}x\right)\mathrm{d}x$

$+\int_{1}^{4}\left(-\frac{1}{15}x^3+\frac{4}{15}x^2\right)\mathrm{d}x$

$\mathrm{E}(X)=-\frac{2}{15}+\frac{27}{20}$ (A1) for correct values

$\mathrm{E}(X)=\frac{73}{60}$ A1

[3]

(b) By considering the graphs of

$y=\frac{1}{20}x+\frac{3}{20}$, $-3\le x\le 1$ and

$y=-\frac{1}{15}x^2+\frac{4}{15}x$, $1<x\le 4$, the maximum point

is $\left(2,\frac{4}{15}\right)$. (M1) for valid approach

Thus, the mode of X is 2. A1

[2]

(c) The standard deviation of X

$=\sqrt{\mathrm{E}(X^2)-(\mathrm{E}(X))^2}$ (M1) for valid approach

$=\sqrt{\int_{-3}^{1} x^2\cdot\left(\frac{1}{20}x+\frac{3}{20}\right)\mathrm{d}x+\int_{1}^{4} x^2\cdot\left(-\frac{1}{15}x^2+\frac{4}{15}x\right)\mathrm{d}x-\left(\frac{73}{60}\right)^2}$ A1

$=\sqrt{2.279722222}$

$=1.509874903$

$=1.51$ A1

[3]

(d) $\int_{-3}^{a}\left(\frac{1}{20}x+\frac{3}{20}\right)\mathrm{d}x=0.35$ A1

$\left[\frac{1}{40}x^2+\frac{3}{20}x\right]_{-3}^{a}=0.35$

$\left(\frac{1}{40}a^2+\frac{3}{20}a\right)-\left(-\frac{9}{40}\right)=0.35$

$a^2+6a+9=14$

$a^2+6a-5=0$ A1

By considering the graph of $y=a^2+6a-5$,

$a=-6.741657387$ (*Rejected*) or $a=0.741657386$.

Thus, the 35th percentile is 0.742. A1

[3]

(e) $\mathrm{P}(|X|>2)=\mathrm{P}(X>2 \text{ or } X<-2)$ (M1) for valid approach

$\mathrm{P}(|X|>2)=\mathrm{P}(X<-2)+\mathrm{P}(X>2)$

$\mathrm{P}(|X|>2)=\int_{-3}^{-2}\left(\frac{1}{20}x+\frac{3}{20}\right)\mathrm{d}x$

$+\int_{2}^{4}\left(-\frac{1}{15}x^2+\frac{4}{15}x\right)\mathrm{d}x$

$\mathrm{P}(|X|>2)=\frac{1}{40}+\frac{16}{45}$ (A1) for correct values

$\mathrm{P}(|X|>2)=\frac{137}{360}$ A1

[3]

(f) $\mathrm{P}(X>2\,|\,|X|>2)=\frac{\mathrm{P}(X>2\cap|X|>2)}{\mathrm{P}(|X|>2)}$ (M1) for valid approach

$\mathrm{P}(X>2\,|\,|X|>2)=\frac{\mathrm{P}(X>2)}{\mathrm{P}(|X|>2)}$

$\mathrm{P}(X>2\,|\,|X|>2)=\frac{\int_{2}^{4}\left(-\frac{1}{15}x^2+\frac{4}{15}x\right)\mathrm{d}x}{\frac{137}{360}}$ A1

$\mathrm{P}(X>2\,|\,|X|>2)=\frac{360}{137}\left(\frac{16}{45}\right)$

$\mathrm{P}(X>2\,|\,|X|>2)=\frac{128}{137}$ A1

[3]